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Superspace Formulation in a Three-Algebra Approach to $D = 3, \mathcal{N} = 4, 5$ Superconformal Chern-Simons Matter Theories

Fa-Min Chen , Yong-Shi Wu

Department of Physics and Astronomy, University of Utah
Salt Lake City, UT 84112-0830, USA
E-mail: fchen@physics.utah.edu, wu@physics.utah.edu

Abstract: We present a superspace formulation of the $D = 3, \mathcal{N} = 4, 5$ superconformal Chern-Simons Matter theories, with matter supermultiplets valued in a symplectic 3-algebra. We first construct an $\mathcal{N} = 1$ superconformal action, and then generalize a method used by Gaitto and Witten to enhance the supersymmetry from $\mathcal{N} = 1$ to $\mathcal{N} = 5$. By decomposing the $\mathcal{N} = 5$ supermultiplets and the symplectic 3-algebra properly and proposing a new super-potential term, we construct the $\mathcal{N}$=4 superconformal Chern-Simons matter theories in terms of two sets of generators of a (quaternion) symplectic 3-algebra. The $\mathcal{N}$=4 theories can also be derived by requiring that the supersymmetry transformations are closed on-shell. The relationship between the 3-algebras, Lie superalgebras, Lie algebras and embedding tensors (proposed in [E. A. Bergshoeff, O. Hohm, D. Roest, H. Samtleben, and E. Sezgin, J. High Energy Phys. 09 (2008) 101.]) is also clarified. The general $\mathcal{N} = 4, 5$ superconformal Chern-Simons matter theories in terms of ordinary Lie algebras can be rederived in our 3-algebra approach. All known $\mathcal{N} = 4, 5$ superconformal Chern-Simons matter theories can be recovered in the present superspace formulation for super-Lie-algebra realization of symplectic 3-algebras.

Keywords: Symplectic 3-Algebras, Superspace, Chern-Simons Matter Theories, M2 branes.

Contents

1. Introduction

In the last two years, extended ($\mathcal{N}\geq 4$) supersymmetric Chern-Simons-matter (CSM) theories in 3D have attracted a lot of interests in the string/M-theory community, because they are natural candidates of the dual gauge theories of multi M2-branes in $M-$theory. Less extended supersymmetric ($\mathcal{N}<4$) CSM theories with arbitrary gauge groups were constructed and investigated long time ago [1]-[6]. Generic Chern-Simons gauge theories

with or without (massless) matter were demonstrated to be conformally invariant even at the quantum level [7, 8, 9, 10, 11]. However, it was much more difficult until recently to construct $\mathcal{N} \geq 4$ CSM theories, since only some special gauge groups are allowed in these theories.

By virtue of the Nambu 3-algebra structure [12, 13], the maximally supersymmetric $\mathcal{N} = 8$ CSM theory with $SO(4)$ gauge group was first constructed independently by Bagger and Lambert [14] and by Gustavsson [15] (BLG). The BLG theory was conjectured to be the dual gauge theory of two M2-branes [16, 17, 18, 19]. The Nambu 3-algebra, equipped with a symmetric and positive-definite metric, has the limitation that it can only generate an $SO(4)$ gauge theory [20, 21, 22], too restrictive for a low-energy effective description of M2-branes.

Very soon Aharony, Bergman, Jafferis and Maldacena (ABJM) have observed [23] that an $\mathcal{N} = 2$ superconformal CSM theory, with gauge group $U(N) \times U(N)$, actually has an $SU(4)$ R-symmetry, hence an enhanced supersymmetry $\mathcal{N} = 6$. The same theory was also obtained by taking the infrared limit of a brane construction. In their formulation, the Nambu 3-algebra structure did not play any role, though the ABJM theory with $SU(2) \times SU(2)$ gauge group is equivalent to the BLG theory. Based on the brane construction, ABJM conjectured that at level k their theory describes the low energy limit of N M2-branes probing a $\mathbf{C}^4/\mathbf{Z}_k$ singularity. In the special cases of $k = 1, 2$, the theory has the maximal supersymmetries ($\mathcal{N} = 8$) [23, 24, 25, 26]. In a large-N limit the ABJM theory is then dual to $M-$theory on $AdS_4 \times S^7/\mathbf{Z}_k$ [23]. The superspace formulation and a manifest $SU(4)$ R-symmetry formulation of the ABJM theory can be found in Ref. [27] and [28], respectively.

In Ref. [29, 30], some extended superconformal gauge theories are constructed by taking a conformal limit of $D = 3$ gauged supergravity theories. In this approach, the embedding tensors play a crucial role. Gaiotto and Witten (GW) [31] have been able to construct a large class of $\mathcal{N} = 4$ CSM theories by a method that enhances $\mathcal{N} = 1$ supersymmetry to $\mathcal{N} = 4$. They also demonstrated that the gauge groups can be classified by super Lie algebras. In Ref. [32], the GW theory was extended to include additional twisted hyper-multiplets; in particular, the extended GW theory with $SO(4)$ gauge group was demonstrated to be equivalent to the BLG theory. In Ref. [33], two new theories, $\mathcal{N} = 5$, $Sp(2M) \times O(N)$ and $\mathcal{N} = 6$, $Sp(2M) \times O(2)$ CSM theories, were constructed by further enhancing the R-symmetry to $Sp(4)$ and $SU(4)$, respectively, and the $\mathcal{N} = 6$, $U(M) \times U(N)$ CSM theory was rederived. The gravity duals of $\mathcal{N} = 5$, $Sp(2M) \times O(N)$ and $\mathcal{N} = 6$, $U(M) \times U(N)$ theories were studied in Ref. [34]. By using group representation theory and applying GW's super-Lie-algebra method for classifying gauge groups, the $\mathcal{N} = 1$ to $\mathcal{N} = 8$ CSM theories were constructed systematically in a recent paper [35].

The progress mentioned in the last two paragraphs was made using mainly ordinary Lie algebras. On the other hand, Bagger and Lambert have been able to construct the $\mathcal{N} = 6, U(M) \times U(N)$ theory in terms of a modified 3-algebra [36]. Unlike the Nambu 3-algebra with totally antisymmetric structure constants, the structure constants of the modified 3-algebra are antisymmetric only in the first two indices. By introducing an invariant antisymmetric tensor into a 3-algebra, hence called a 'symplectic 3-algebra', another class

of $\mathcal{N}=6$ CSM theories, with gauge group $Sp(2M)\times O(2)$, has been constructed by the authors of the present paper [37]. We have also demonstrated that the $\mathcal{N}=6, U(M)\times U(N)$ theory can be recast into the symplectic 3-algebraic formalism [37]. In Ref. [38], both the general $\mathcal{N}=5$ and $\mathcal{N}=6$ CSM theories have been formulated in a unified symplectic 3-algebraic framework. These theories based on 3-algebras were constructed by requiring that the supersymmetries must be closed on-shell. All examples of $\mathcal{N}=5$ theories were recovered in Ref. [39] by specifying the 3-algebra structure constants.

One goal of the present paper is to combine the superspace formalism with the 3-algebra, to rederive the $\mathcal{N}=5$ theories by using the Giatto-Witten enhancement mechanism. Previously the $\mathcal{N}=5$ theories were derived from the $\mathcal{N}=4$ theories by carefully choosing the gauge groups [33, 35]. So the construction of $\mathcal{N}=5$ theories by enhancing $\mathcal{N}=1$ supersymmetry is interesting in its own right, especially in a 3-algebraic framework. It provides insight into the relationship between the 3-algebra and conventional Lie-algebra approach.

Another goal is to construct general $\mathcal{N}=4$ theories in the (quaternion) 3-algebra framework, in which there are two similar sets of complex 3-algebra generators. These $\mathcal{N}=4$ theories are 3-algebra version of Chern-Simons quiver gauge theories. We will construct the $\mathcal{N}=4$ theories by two distinct methods. We first start from $\mathcal{N}=5$ supermultiplets, decompose them and the symplectic 3-algebra generators properly, and propose a new superpotential which is $\mathcal{N}$=4 superconformally invariant. Alternatively, we will derive the same $\mathcal{N}$=4 theories by requiring that the supersymmetry transformations are closed on-shell, i.e., we will examine the $\mathcal{N}=4$ algebra and check its closure. The closure of $\mathcal{N}=4$ algebra in the GW theory (*without* the twisted hypermultiplets) has been checked in Ref. [31]. However, to our knowledge, the closure of the algebra in theories *with* the twisted hypermultiplets has not been explicitly checked in the literature. So our calculation will fill this gap.

We will systematically investigate the relations between the 3-algebras, Lie superalgebras, ordinary Lie algebras and embedding tensors that are used to build $D=3$ extended supergravity theories in Ref. [30]. The relations between the 3-algebras and Lie superalgebras are explored in Ref. [35, 40, 41], using representation theory. They did not discuss the relations between the embedding tensors in Ref. [30] and 3-algebras or Lie superalgebras. We will fill this gap by a more physical approach.

We will demonstrate that the symplectic 3-algebra can be realized in terms of a super Lie algebra. The generators of the 3-algebra T_I can be realized as the fermionic generators of the super Lie algebra Q_I, and the 3-bracket is realized in terms of a double graded bracket: $[T_I, T_J; T_K] \doteq [\{Q_I, Q_J\}, Q_K]$. In this realization, the fundamental identity (FI) of the symplectic 3-algebra can be converted into the MMQ Jacobi identity of the super Lie algebra (M is a bosonic generator). It will be shown that the structure constants of the symplectic 3-algebra furnish a quaternion representation of the bosonic part of the super Lie algebra, and play the role of Killing-Cartan metric of the bosonic part of the super Lie algebra. Then the FI of the 3-algebra is rewritten as an ordinary commutator, whose structure constants are totally antisymmetric. Moreover, we prove that the structure constants of the symplectic 3-algebra are the components of the embedding tensor proposed

in [30], if we realize the symplectic 3-algebra in terms of the super Lie algebra.

The general $\mathcal{N}=4,5$ superconformal Chern-Simons-matter theories in terms of ordinary Lie algebras can be re-drived from our super-Lie-algebra realization of the symplectic 3-algebras. Not only all known examples of $\mathcal{N}=4,5$ ordinary CSM theories, but also $\mathcal{N}=4$ CSM quiver gauge theories (including some new examples), can be produced as well. The details for the latter will be presented in a forthcoming paper. Therefore, our superspace formulation for the super-Lie-algebra realization of symplectic 3-algebras provide a unified treatment of all known $\mathcal{N}=4,5,6,8$ CSM theories, including new examples of $\mathcal{N}=4$ quiver gauge theories as well.

This paper is organized as follows. In Sec. 2.1 we review symplectic 3-algebras and define the notations. Sec. 2.2 is devoted to the construction of the $\mathcal{N}=5$ theories by enhancing the supersymmetry from $\mathcal{N}=1$ to $\mathcal{N}=5$ in a 3-algebraic framework. In Sec. 3.1, we derive the $\mathcal{N}$=4 theories by decomposing the $\mathcal{N}=5$ supermultiplets and the symplectic 3-algebra properly and proposing a new superpotential. The closure of the $\mathcal{N}=4$ algebra is explicitly verified in Sec. 3.2. In Sec. 4, we discuss the relations between 3-algebras, super Lie algebras, ordinary Lie algebras and the embedding tensors proposed in Ref. [30]. In Sec. 5, we present how to reproduce the Lie algebra version of $\mathcal{N}=4,5$ theories from the 3-algebra approach. The last Sec. is devoted to conclusions.

2. $\mathcal{N}=5$ theories and Symplectic Three-Algebras

2.1 A Review of Symplectic Three-Algebra

A 3-algebra is a complex vector space equipped a 3-bracket, mapping three vectors to one vector [38]:

$$[T_I, T_J; T_K] = f_{IJK}{}^{L} T_L, \tag{2.1}$$

where T_I $(I=1,2,\ldots,M)$ is a set of generators. The set of complex numbers $f_{IJK}{}^{L}$ are called the structure constants. We define the global transformation of a field X valued in this 3-algebra ($X = X^K T_K$) as [14]:

$$\delta_{\tilde{\Lambda}} X = \Lambda^{IJ}[T_I, T_J; X], \tag{2.2}$$

where the parameter Λ^{IJ} is independent of spacetime coordinate. (The symmetry transformation (2.2) will be gauged later). For (2.2) to a symmetry, one has to require that it acts as a derivative [14]:

$$\delta_{\tilde{\Lambda}}([X,Y;Z]) = [\delta_{\tilde{\Lambda}} X, Y; Z] + [X, \delta_{\tilde{\Lambda}} Y; Z] + [X, Y; \delta_{\tilde{\Lambda}} Z], \tag{2.3}$$

where $Y = Y^N T_N$ and $Z = Z^K T_K$. Canceling Λ^{IJ}, X^M, Y^N and Z^K from both sides, we obtain the following FI satisfied by the generators:

$$[T_I, T_J; [T_M, T_N; T_K]] = [[T_I, T_J; T_M], T_N; T_K] + [T_M, [T_I, T_J; T_N]; T_K] + [T_M, T_N; [T_I, T_J; T_K]]. \tag{2.4}$$

The FI is a generalization of the Jacobi identity of an ordinary Lie algebra. Combining the three-bracket (2.1) and the FI (2.4), we find that the FI satisfied by the structure constants is

$$f_{MNK}{}^{O} f_{IJO}{}^{L} = f_{IJM}{}^{O} f_{ONK}{}^{L} + f_{IJN}{}^{O} f_{MOK}{}^{L} + f_{IJK}{}^{O} f_{MNO}{}^{L}. \tag{2.5}$$

To define a symplectic 3-algebra, we introduce a symplectic bilinear form into the 3-algebra:

$$\omega(X,Y) = \omega_{IJ} X^I Y^J. \tag{2.6}$$

We denote the inverse of the antisymmetric tensor ω_{IJ} as ω^{IJ}. The existence of the inverse implies that a 3-algebra index I must run from 1 to $M = 2L$. We will use ω_{IJ} and ω^{IJ} to lower or raise 3-algebra indices; for instance, $f_{IJKL} \equiv \omega_{LM} f_{IJK}{}^{M}$. The symplectic bilinear form must be invariant under an arbitrary global transformation:

$$\begin{aligned} \delta_{\tilde{\Lambda}}(\omega_{IJ} X^I Y^J) &= \Lambda^{LM}(f_{LMI}{}^{K}\omega_{KJ} + f_{LMJ}{}^{K}\omega_{IK}) X^I Y^J \\ &= 0. \end{aligned} \tag{2.7}$$

It turns out that the structure constants must be *symmetric* in the last two indices:

$$f_{LMIJ} = f_{LMJI}. \tag{2.8}$$

From point of view of ordinary Lie group, the infinitesimal matrices

$$\tilde{\Lambda}^{K}{}_{I} \equiv \Lambda^{LM} f_{LM}{}^{K}{}_{I} \tag{2.9}$$

must form the Lie algebra $Sp(2L, \mathbb{C})$. We call the 3-algebra defined by the above equations a symplectic 3-algebra.

Since the 3-algebra is also a complex vector space, one can define a Hermitian bilinear form

$$h(X,Y) = X^{*I} Y^I \tag{2.10}$$

(with X^{*I} the complex conjugate of X^I), which is naturally positive-definite and will be used to construct the Lagrangians. The Hermitian bilinear form is also required to be invariant under the global transformation:

$$\begin{aligned} \delta_{\tilde{\Lambda}}(X^{*I} Y^I) &= (\Lambda^{*LM} f^{*}_{LMI}{}^{K} + \Lambda^{LM} f_{LMK}{}^{I}) X^{*I} Y^K \\ &= 0. \end{aligned} \tag{2.11}$$

To solve the above equation, we assume that the parameter Λ^{LM} is Hermitian: $\Lambda^{*LM} = \Lambda_{ML}$. Since it also carries two symplectic 3-algebra indices, it obeys the natural reality condition $\Lambda^{*LM} = \omega_{LI}\omega_{MJ}\Lambda^{IJ}$. These two equations imply that the parameter is symmetric, i.e. $\Lambda_{ML} = \Lambda_{LM}$. In summary, we have

$$\Lambda^{*LM} = \Lambda_{ML} = \Lambda_{LM}. \tag{2.12}$$

Now since the parameter Λ^{IJ} is symmetric, re-examining the global transformation (2.2) leads us to require that the structure constants are symmetric in the first two indices:

$$f_{IJKL} = f_{JIKL}. \tag{2.13}$$

With Eq. (2.12) and (2.13), we find that Eq. (2.11) can be satisfied if we impose the following reality condition on the structure constants:

$$f^*_{LMIK} = f^{MLKI} \quad \text{or} \quad f^{*L}{}_M{}^I{}_K = f^M{}_L{}^K{}_I. \tag{2.14}$$

Now both the symplectic bilinear form (2.6) and the Hermitian bilinear (2.10) form are invariant under the global transformation (2.2). So from point of view of ordinary Lie group, the symmetry group generated by the 3-algebra transformations (2.2) is nothing but $Sp(2L)$, which is the intersection of $U(2L)$ and $Sp(2L,\mathbb{C})$.

Later we will see, to enhance the super-symmetry from $\mathcal{N}=1$ to $\mathcal{N}=5$, we will require the 3-bracket to satisfy an additional constraint condition:

$$\omega([T_I, T_{(J}; T_K], T_{L)}) = 0, \tag{2.15}$$

or simply $f_{I(JKL)} = 0$. Combining Eq. (2.15) with (2.8) and (2.13), we have that $f_{(IJK)L} = 0$ and $f_{IJKL} = f_{KLIJ}$. In summary, the structure constants f_{IJKL} enjoy the symmetry properties

$$f_{IJKL} = f_{JIKL} = f_{JILK} = f_{KLIJ}. \tag{2.16}$$

2.2 $\mathcal{N}=5$ Theories in Terms of 3-Algebras

In this subsection, we will generalize Giaotto and Witten's idea and method [31] to enhance the super-symmetry from $\mathcal{N}=1$ to $\mathcal{N}=5$. [1] We will work in a three-algebraic framework.

Let us first explain the mechanism for supersymmery enhancement. We assume that the $\mathcal{N}=1$ superfields for the matter fields are 3-algebra valued (our notation and convention are summarized in appendix A):

$$\Phi^I_A = Z^I_A + i\theta\gamma_A{}^B\psi^I_B - \frac{i}{2}\theta^2 F^I_A, \tag{2.17}$$

where I is a 3-algebra index, A, B are $Sp(4) \cong SO(5)$ indices $(A, B = 1, ..., 4)$, and $\gamma_A{}^B$ is a Hermitian $SO(5) \equiv Sp(4)$ gamma matrix, satisfying $\gamma_A{}^B\gamma_B{}^C = \delta_A{}^C$. [2] The superfield Φ satisfies the reality condition:

$$\bar\Phi^A_I = \Phi^{\dagger I}_A = \omega^{AB}\omega_{IJ}\Phi^J_B. \tag{2.18}$$

The purpose for introducing the gamma matrix into the second term of (2.17) is the following: after we promote the supersymmetry from $\mathcal{N}=1$ to $\mathcal{N}=5$, we want the supercharges and the matter fields to transform as the **5** and **4** of $Sp(4)$, respectively, with the gamma matrix being the couplings.

Despite that Φ^I_A carries an $Sp(4)$ index, it is still an $\mathcal{N}=1$ superfields in that it just depends on one copy of fermionic coordinates θ^α. Generally speaking, if we use (2.17) to

[1] In Ref. [42], the $\mathcal{N}=8$ BLG theory was constructed by using $\mathcal{N}=1$ superspace formulation in the Nambu 3-algebra approach.

[2] Generally $\gamma_A{}^B \equiv c_m\gamma^{mB}_A$ $(m = 1, ..., 5)$, where γ^{mB}_A are the $SO(5)$ gamma matrices (see appendix A.4), and c_m real coefficients. We normalize the parameters c_m so that $\delta^{mn}c_mc_n = 1$. The non-uniqueness of this gamma matrix is exactly what are allowed by the R-symmetry $SO(5)$.

construct an $\mathcal{N}=1$ CSM theory, the Yukawa couplings will contain the gamma matrix ${\gamma_A}^B$, which is not $Sp(4)$ invariant. [3] As a result, the CSM theory is generally not $Sp(4)$ invariant. However, we are be able to remove the gamma matrix ${\gamma_A}^B$ from the theory by adjusting the superspace couplings. The resulting theory then have an $Sp(4)$ global symmetry, which does *not* commute with the $\mathcal{N}=1$ supersymmetry. Namely the supercharge transforms non-trivially under the $Sp(4)$ global symmetry group. More precisely, the supercharges transform in the vector representation of $SO(5)$ or **5** of $Sp(4)$. As a result, the supersymmetry gets enhanced from $\mathcal{N}=1$ to $\mathcal{N}=5$. We will explain this point in details when we examine the supersymmetry transformations.

To construct the $\mathcal{N}=1$ CSM theory, we first gauge the global symmetry transformation (2.2). We define the gauge transformation of the superfield Φ^I as

$$\delta_{\tilde{\Lambda}}\Phi^I_A = \Lambda^{KL}{f_{KL}}^I{}_J\Phi^J_A = \tilde{\Lambda}^I{}_J\Phi^J_A, \tag{2.19}$$

where the parameter Λ^{KL} is a superfield, depending on the coordinates of the superspace. We then define the covariant derivatives as

$$(D_\alpha)^I{}_J = \mathscr{D}_\alpha\delta^I{}_J + {\tilde{\Gamma}_\alpha}{}^I{}_J \quad \text{and} \quad (D_\mu)^I{}_J = \partial_\mu\delta^I{}_J + {\tilde{\Gamma}_\mu}{}^I{}_J, \tag{2.20}$$

where $\mathscr{D}_\alpha$ is the super-covariant derivative, defined by Eq. (A.9). In accordance with our basic definition (2.2), it is natural to assume that the super-connections take the following forms

$$\tilde{\Gamma}_\alpha{}^I{}_J \equiv \Gamma^{KL}_\alpha {f_{KL}}^I{}_J \quad \text{and} \quad \tilde{\Gamma}_\mu{}^I{}_J \equiv \Gamma^{KL}_\mu {f_{KL}}^I{}_J, \tag{2.21}$$

transforming as [4]

$$\delta_{\tilde{\Lambda}}\tilde{\Gamma}_\alpha{}^I{}_J = -D_\alpha\tilde{\Lambda}^I{}_J \quad \text{and} \quad \delta_{\tilde{\Lambda}}\tilde{\Gamma}_\mu{}^{\mathrm{I}}{}_{\mathrm{J}} = -\mathrm{D}_\mu\tilde{\Lambda}^{\mathrm{I}}{}_{\mathrm{J}}, \tag{2.22}$$

respectively. In the Wess-Zumino gauge, the super-connection $\tilde{\Gamma}_\alpha$ takes the form

$$\begin{aligned}\tilde{\Gamma}_\alpha{}^I{}_J &= i\theta^\beta\tilde{A}_{\alpha\beta}{}^I{}_J + \theta^2\tilde{\chi}_\alpha{}^I{}_J \\ &= (i\theta^\beta A^{KL}_{\alpha\beta} + \theta^2\chi^{KL}_\alpha){f_{KL}}^I{}_J,\end{aligned} \tag{2.23}$$

where $\tilde{\chi}_\alpha{}^I{}_J$ is superpartner of the gauge field $\tilde{A}_{\alpha\beta}{}^I{}_J$. In accordance with (2.12), we assume that $A^{KL}_{\alpha\beta}$ and χ^{KL}_α are Hermitian and symmetric in KL. The two superconnections (2.21) should not be independent, since there is only one gauge symmetry. Actually, imposing the conventional constraint [43]

$$\{D_\alpha, D_\beta\} = 2iD_{\alpha\beta} \tag{2.24}$$

determines the vector superconnection:

$$\tilde{\Gamma}_{\alpha\beta}{}^I{}_J = \tilde{A}_{\alpha\beta}{}^I{}_J - i\theta_\alpha\tilde{\chi}_\beta{}^I{}_J - i\theta_\beta\tilde{\chi}_\alpha{}^I{}_J + \frac{i}{2}\theta^2\tilde{F}_{\alpha\beta}{}^I{}_J, \tag{2.25}$$

[3] With the standard definition ${\Sigma_A}^B \equiv \frac{1}{2}\omega_{mn}\Sigma^{mnB}_A$, where $\Sigma^{mn} = \frac{1}{4}[\gamma^m, \gamma^n]$, we note that

$$\delta{\gamma_A}^B \equiv {\Sigma_A}^C{\gamma_C}^B - {\Sigma_C}^B{\gamma_A}^C = \omega_{mn}c^n\gamma^{mB}_A.$$

Thus, ${\gamma_A}^B$ is *not* $Sp(4)$ invariant.

[4] In this section, we define a general tilde field $\tilde{\Psi}$ as $\tilde{\Psi}^I{}_J \equiv \Psi^{KL}{f_{KL}}^I{}_J$, where Ψ^{KL} can be a superfield or an ordinary field.

where the field strength is defined as

$$\tilde{F}_{\alpha\beta}{}^{I}{}_{J} = \frac{1}{2}(\partial_\alpha{}^\gamma \tilde{A}_{\gamma\beta}{}^{I}{}_{J} + \partial_\beta{}^\gamma \tilde{A}_{\gamma\alpha}{}^{I}{}_{J}) + \frac{1}{2}[\tilde{A}_\alpha{}^\gamma, \tilde{A}_{\gamma\beta}]^{I}{}_{J}; \quad \tilde{F}_{\mu\nu}{}^{I}{}_{J} = \frac{1}{2}(\gamma_{\mu\nu})^{\alpha\beta}\tilde{F}_{\alpha\beta}{}^{I}{}_{J}. \tag{2.26}$$

The superfield $\Gamma^{KL}_\mu = -\frac{1}{2}\gamma^{\alpha\beta}_\mu \Gamma^{KL}_{\alpha\beta}$ in Eq. (2.21) can be read off from Eq. (2.25) by re-writing the field strength as a product of a field and the structure constants:

$$\begin{aligned}\tilde{F}_{\alpha\beta}{}^{I}{}_{J} &= \frac{1}{2}[\partial_\alpha{}^\gamma A^{KL}_{\gamma\beta} + \partial_\beta{}^\gamma A^{KL}_{\gamma\alpha} + (\tilde{A}_\alpha{}^\gamma)^L{}_M A^{MK}_{\gamma\beta} + (\tilde{A}_\beta{}^\gamma)^K{}_M A^{ML}_{\gamma\alpha}] f_{KL}{}^{I}{}_{J} \\ &\equiv F^{KL}_{\alpha\beta} f_{KL}{}^{I}{}_{J}.\end{aligned} \tag{2.27}$$

In the first line we have used the FI (2.5).

To be self-consistent, the covariant derivative D_α must satisfy the Jacobi identity:

$$[D_\alpha, \{D_\beta, D_\gamma\}] + [D_\beta, \{D_\gamma, D_\alpha\}] + [D_\gamma, \{D_\alpha, D_\beta\}] = 0. \tag{2.28}$$

The Jacobi identity can be solved by introducing a superfield strength $\tilde{\mathcal{W}}_\alpha$ [43]:

$$[D_\alpha, D_{\beta\gamma}] = i\epsilon_{\alpha\beta}\tilde{\mathcal{W}}_\gamma + i\epsilon_{\alpha\gamma}\tilde{\mathcal{W}}_\beta. \tag{2.29}$$

By direct calculation, we obtain

$$\begin{aligned}\tilde{\mathcal{W}}_\alpha{}^{I}{}_{J} &= \tilde{\chi}_\alpha{}^{I}{}_{J} + \theta^\beta \tilde{F}_{\alpha\beta}{}^{I}{}_{J} - \frac{i}{2}\theta^2 (D_\alpha{}^\beta \tilde{\chi}_\beta)^{I}{}_{J} \\ &= [\chi^{KL}_\alpha + \theta^\beta F^{KL}_{\alpha\beta} - \frac{i}{2}\theta^2 (D_\alpha{}^\beta \chi_\beta)^{KL}] f_{KL}{}^{I}{}_{J} \\ &\equiv \mathcal{W}^{KL}_\alpha f_{KL}{}^{I}{}_{J},\end{aligned} \tag{2.30}$$

with

$$(D_\alpha{}^\beta \chi_\beta)^{KL} f_{KL}{}^{I}{}_{J} \equiv [\partial_\alpha{}^\beta \chi^{KL}_\beta + (\tilde{A}_\alpha{}^\beta)^L{}_M \chi^{MK}_\beta + (\tilde{A}_\alpha{}^\beta)^K{}_M \chi^{MJ}_\beta] f_{KL}{}^{I}{}_{J}. \tag{2.31}$$

In deriving the above equation, we have used the FI (2.5) again. Here we would like to make one comment on the relation between the FI (2.5) and the anti-commutator (2.24) and the Jacobi identity (2.28). Without consulting the FI, one would not be able to derive Eq. (2.27) and write $\tilde{\Gamma}_{\alpha\beta}{}^{I}{}_{J}$ as $\Gamma^{KL}_{\alpha\beta} f_{KL}{}^{I}{}_{J}$. This would be inconsistent with our assumption (2.21) or the basic definition (2.2). Similarly, the superfield strength would not take the form $\tilde{\mathcal{W}}_\alpha{}^{I}{}_{J} = \mathcal{W}^{KL}_\alpha f_{KL}{}^{I}{}_{J}$ without the FI (see Eq. (2.30)). Recall that the vector superconnection and the superfield strength are defined through (2.24) and (2.28), respectively. So, had we not introduce the FI in Sec. 2.1, we would have to introduce the FI in this subsection for making the 3-bracket (2.2) consistent with (2.24) and (2.28).

After gauging the symmetry (2.2) in the superspace, we are ready to construct an $\mathcal{N} = 1$ CSM theory. A general $\mathcal{N} = 1$ CSM theory consists of three parts: $\mathcal{L} = \mathcal{L}_{\rm kin} + \mathcal{L}_{\rm CS} + \mathcal{L}_W$, where $\mathcal{L}_{\rm kin}$ is the Lagrangian of the kinetic terms of the matter fields, $\mathcal{L}_{\rm CS}$ the Chern-Simons term and $\mathcal{L}_W$ the superpotential. The first part $\mathcal{L}_{\rm kin}$ is standard:

$$\begin{aligned}\mathcal{L}_{\rm kin} &= \frac{1}{8}\int d^2\theta D^\alpha \bar{\Phi}^A_I D_\alpha \Phi^I_A \\ &= \frac{1}{2}(-D_\mu \bar{Z}^A_I D^\mu Z^I_A + i\bar{\psi}^A_I \gamma_\mu D^\mu \psi^I_A + 2i f_{IJKL}\gamma_B{}^A \bar{\psi}^{BK}\chi^{IJ} Z^L_A + \bar{F}^A_I F^I_A).\end{aligned} \tag{2.32}$$

The covariant derivatives are given by

$$D_\mu Z_I^A = \partial_\mu Z_I^A - \tilde{A}_\mu{}^J{}_I Z_J^A, \tag{2.33}$$

$$D_\mu Z_A^I = \partial_\mu Z_A^I + \tilde{A}_\mu{}^I{}_J Z_A^J. \tag{2.34}$$

We propose the Chern-Simons term as

$$\begin{aligned}\mathcal{L}_{\rm CS} &= \frac{1}{8}\int d^2\theta[-if_{IJKL}\Gamma^{\alpha IJ}\mathcal{W}_\alpha^{KL} + \frac{1}{3}f_{IJK}{}^O f_{OLMN}\Gamma^{\alpha IJ}\Gamma^{\beta KL}\Gamma_{\alpha\beta}^{MN}] \\ &= \frac{1}{2}\epsilon^{\mu\nu\lambda}(f_{IJKL}A_\mu^{IJ}\partial_\nu A_\lambda^{KL} + \frac{2}{3}f_{IJK}{}^O f_{OLMN}A_\mu^{IJ}A_\nu^{KL}A_\lambda^{MN}) + \frac{i}{2}f_{IJKL}\chi^{\alpha IJ}\chi_\alpha^{KL}.\end{aligned} \tag{2.35}$$

The first part of the second line is precisely the 'twisted' Chern-Simons term in Ref. [38], while the gaugino χ is just an auxiliary field, whose equation of motion is

$$\chi^{\alpha IJ} = -\gamma_B{}^A\psi^{\alpha B(I}Z_A^{J)}. \tag{2.36}$$

Substituting it into (2.32) and (2.35) gives a Yukawa coupling:

$$-\frac{i}{2}Z_A^I Z_B^J\psi_C^K\psi_D^L f_{IKJL}\gamma^{AC}\gamma^{BD}. \tag{2.37}$$

Note that this term is not $Sp(4)$ invariant, because the gamma matrix is *not* $Sp(4)$ invariant (see footnote 2).

Let us now consider the superpotential $W(\Phi)$. It must satisfy two conditions. First, for conformal invariance, the superpotential must be homogeneous and quartic in Φ; schematically, $W(\Phi) \sim \Phi\Phi\Phi\Phi$. Second, after combining (2.37) with the Yukawa terms arising from $W(\Phi)$, the final expression must be $Sp(4)$ invariant. Before proposing $W(\Phi)$, it is useful to look at the structure of (2.37): it contains $\gamma^{AC}\gamma^{BD}$. The essential observation is that $\gamma^{[AC}\gamma^{BD]}$ has to be proportional to the totally antisymmetric (invariant) tensor ε^{ABCD}, since this tensor is unique in $Sp(4)$. The precise expression is

$$\begin{aligned}-\varepsilon^{ABCD} &= \gamma^{AC}\gamma^{BD} - \gamma^{BC}\gamma^{AD} + \gamma^{BA}\gamma^{CD} \\ &= \omega^{AB}\omega^{CD} - \omega^{AC}\omega^{BD} + \omega^{AD}\omega^{BC}.\end{aligned} \tag{2.38}$$

Namely, our problem may be solved if the final expression for (2.37) plus the Yukawa terms arising from $W(\Phi)$ is somehow related to (2.38). So we are inspired to propose the following superpotential

$$W(\Phi) = \frac{1}{4}(g_{IJKL}\omega^{AB}\omega^{CD}\Phi_A^I\Phi_B^J\Phi_C^K\Phi_D^L + \tilde{g}_{IJKL}\gamma^{AB}\gamma^{CD}\Phi_A^I\Phi_B^J\Phi_C^K\Phi_D^L), \tag{2.39}$$

where the 3-algebra tensor g satisfies $g_{IJKL} = -g_{JIKL} = -g_{IJLK} = g_{KLIJ}$, and $\tilde{g}$ has the same symmetry properties. We require that the tensors g and $\tilde{g}$ are gauge invariant. This implies that g and $\tilde{g}$ can be expressed in terms of ω_{IJ} and f_{IJKL}, the only two gauge invariant quantities. After carrying out the Berezin integration $\frac{i}{2}\int d^2\theta W(\Phi)$, we obtain

$$\begin{aligned}\mathcal{L}_W &= -\frac{i}{2}Z_A^I Z_B^J\psi_C^K\psi_D^L(g_{IJKL}\omega^{AB}\omega^{CD} + 2g_{IKJL}\gamma^{AC}\gamma^{BD} + \tilde{g}_{IJKL}\gamma^{AB}\gamma^{CD} + 2\tilde{g}_{IKJL}\omega^{AC}\omega^{BD}) \\ &\quad -(g_{IJKL}\omega^{AB}\omega^{CD} + \tilde{g}_{IJKL}\gamma^{AB}\gamma^{CD})Z_B^J Z_C^K Z_D^L F_A^I.\end{aligned} \tag{2.40}$$

The first and last term of the first line are already $Sp(4)$ invariant. Combining the middle two terms of the first line with (2.37) gives

$$-\frac{i}{2}Z^I_A Z^J_B \psi^K_C \psi^L_D[(2g_{IKJL}+f_{IKJL})\gamma^{AC}\gamma^{BD}+\tilde{g}_{IJKL}\gamma^{AB}\gamma^{CD}]. \tag{2.41}$$

Since we wish to use Eq. (2.38), we first have to anti-symmetrize AB in the expression $\gamma^{AC}\gamma^{BD}$. Equivalently, we have to set the part proportional to $Z^{(I}_{(A}Z^{J)}_{B)}$ to be zero:

$$g_{IKJL}+g_{JKIL}+\frac{1}{2}f_{IKJL}+\frac{1}{2}f_{JKIL}=0. \tag{2.42}$$

Now the remaining part of (2.41) is antisymmetric in AB:

$$\frac{i}{2}Z^I_A Z^J_B \psi^K_C \psi^L_D[(2g_{IKJL}+f_{IKJL})\gamma^{C[A}\gamma^{B]D}-\tilde{g}_{IJKL}\gamma^{AB}\gamma^{CD}]. \tag{2.43}$$

It can be seen that if we set

$$\tilde{g}_{IJKL}=-\frac{1}{2}(g_{IKJL}-g_{JKIL}+\frac{1}{2}f_{IKJL}-\frac{1}{2}f_{JKIL}) \tag{2.44}$$

and apply the key identity (2.38), then Eq. (2.43) becomes

$$\frac{i}{2}Z^I_A Z^J_B \psi^K_C \psi^L_D \tilde{g}_{IJKL}(\omega^{AB}\omega^{CD}-\omega^{AC}\omega^{BD}+\omega^{AD}\omega^{BC}). \tag{2.45}$$

Now Eq. (2.45) is manifestly $Sp(4)$ invariant. However we still need to solve (2.42) and (2.44) in terms of f_{IJKL} and ω_{IJ}. An equation similar to (2.42) is first derived by GW [31]:

$$g_{IKJL}+g_{JKIL}+\frac{3}{4}k_{mn}\tau^m_{IK}\tau^n_{JL}+\frac{3}{4}k_{mn}\tau^m_{JK}\tau^n_{IL}=0, \tag{2.46}$$

where the set of matrices τ^m_{IK} is in the fundamental representation of $Sp(2L)$ or its subalgebra, and k_{mn} is the Killing-Cartan metric. Although the ($\mathcal{N}=4$) GW theory is not an $\mathcal{N}=5$ theory, the similarity between (2.42) and (2.46) strongly suggests that f_{IJKL} can be specified as $k_{mn}\tau^m_{IJ}\tau^n_{KL}$ (up to an unimportant constant). This is indeed the case: the FI (2.5) does admit an explicit solution in terms of the tensor product $f_{IJKL}=k_{mn}\tau^m_{IJ}\tau^n_{KL}$. It is straightforward to verify that $f_{IJKL}=k_{mn}\tau^m_{IJ}\tau^n_{KL}$ satisfy the FI (2.5). This solution is first found by Gustavsson by converting the FI into two independent commutators of ordinary Lie algebra [15]. Later we will discuss the relations between the 3-algebra and the ordinary Lie algebra in details. Eq. (2.42) can be easily solved by adopting a method in Ref. [31]. Summing (2.42) over cyclic permutations of IKJ gives

$$f_{(IKJ)L}=0, \quad \text{or} \quad f_{I(KJL)}=0. \tag{2.47}$$

This is precisely (2.15), as we stated earlier. The above equation is also derived by requiring that the $\mathcal{N}=5$ supersymmetry transformations are closed on-shell [38]. Eq. (2.42) is solved by setting

$$g_{IKJL}=\frac{1}{6}(f_{IJKL}-f_{ILKJ}). \tag{2.48}$$

Substituting (2.48) into (2.44), we obtain

$$\tilde{g}_{IJKL} = \frac{1}{3}(f_{ILJK} - f_{IKJL}). \tag{2.49}$$

Substituting (2.49) into (2.45), then combining (2.45) with the first and the last term of the first line of (2.40), we reach the final expression for all Yukawa terms:

$$-\frac{i}{2}\omega^{AB}\omega^{CD}f_{IJKL}(Z^I_A Z^K_B \psi^J_C \psi^L_D - 2Z^I_A Z^K_D \psi^J_C \psi^L_B). \tag{2.50}$$

Finally we integrate out the auxiliary field F^I_A appearing in (2.32) and (2.40):

$$\bar{F}^A_I = \frac{1}{3}f_{IKLJ}\omega^{BC}\omega^{AD}Z^K_B Z^L_C Z^J_D - \frac{2}{3}f_{IKLJ}\gamma^{BC}\gamma^{AD}Z^K_B Z^L_C Z^J_D. \tag{2.51}$$

Now it is straightforward to calculate the bosonic potential:

$$\begin{aligned} -\frac{1}{2}\bar{F}^A_I F^I_A = {} & \frac{1}{18}f_{IJKO}f^O{}_{LMN}(-\omega^{AC}\omega^{BE}\omega^{DF} + 2\omega^{AC}\gamma^{BE}\gamma^{DF} \\ & +2\omega^{DF}\gamma^{AC}\gamma^{BE} - 4\omega^{BE}\gamma^{AC}\gamma^{DF})Z^I_A Z^J_B Z^K_C Z^L_D Z^M_E Z^N_F. \end{aligned} \tag{2.52}$$

Note that $V = \frac{1}{2}\bar{F}^A_I F^I_A$ is positive definite, though it is not manifestly $Sp(4)$ invariant due to the presence of the gamma matrix. However, by taking advantage of the key identity (2.38) and the constraint condition $f_{(IJK)L} = 0$, we are able to prove that (2.52) is indeed $Sp(4)$ invariant (see appendix B). The final expression for the bosonic potential is

$$V = -\frac{1}{60}(2f_{IJK}{}^O f_{OLMN} - 9f_{KLI}{}^O f_{ONMJ} + 2f_{IJL}{}^O f_{OKMN})Z^N_A Z^{AI} Z^J_B Z^{BK} Z^L_C Z^{CM}. \tag{2.53}$$

In summary, the full Lagrangian in terms of the symplectic 3-algebra is given by

$$\begin{aligned} \mathcal{L} = {} & \frac{1}{2}(-D_\mu \bar{Z}^A_I D^\mu Z^I_A + i\bar{\psi}^A_I \gamma_\mu D^\mu \psi^I_A) \\ & -\frac{i}{2}\omega^{AB}\omega^{CD}f_{IJKL}(Z^I_A Z^K_B \psi^J_C \psi^L_D - 2Z^I_A Z^K_D \psi^J_C \psi^L_B) \\ & +\frac{1}{2}\epsilon^{\mu\nu\lambda}(f_{IJKL}A^{IJ}_\mu \partial_\nu A^{KL}_\lambda + \frac{2}{3}f_{IJK}{}^O f_{OLMN}A^{IJ}_\mu A^{KL}_\nu A^{MN}_\lambda) \\ & +\frac{1}{60}(2f_{IJK}{}^O f_{OLMN} - 9f_{KLI}{}^O f_{ONMJ} + 2f_{IJL}{}^O f_{OKMN})Z^N_A Z^{AI} Z^J_B Z^{BK} Z^L_C Z^{CM}. \end{aligned} \tag{2.54}$$

This Lagrangian is exactly the same as the $\mathcal{N} = 5$ Lagrangian derived by requiring that the supersymmetry transformations are closed on-shell [38]. Using the reality condition (2.14), one can recast the potential term into the following form:

$$V = \frac{2}{15}(\Upsilon^L_{ABC})^*\Upsilon^L_{ABC}, \tag{2.55}$$

where

$$\Upsilon^L_{ABC} \equiv f_{IJK}{}^L(Z^I_A Z^J_B Z^K_C + \frac{1}{4}\omega_{BC}Z^I_A Z^J_D Z^{DK}). \tag{2.56}$$

Now the potential term is manifestly positive definite.

Let us consider the supersymmetry transformations. The $\mathcal{N}=1$ supersymmetry transformation of the scalar field is

$$\delta_Q Z^I_A = i\epsilon^\alpha \gamma_A{}^B \psi^I_{\alpha B}. \tag{2.57}$$

On the other hand, the action (2.54) is invariant under the $Sp(4)$ global symmetry transformation

$$\delta_R Z^I_A = \Sigma_A{}^B Z^I_B \quad , \quad \delta_R \psi^I_A = \Sigma_A{}^B \psi^I_B. \tag{2.58}$$

Therefore one can consider the commutator of δ_R and δ_Q:

$$[\delta_R, \delta_Q] Z^I_A = i\epsilon^\alpha (\gamma_A{}^B \Sigma_B{}^C - \Sigma_A{}^B \gamma_B{}^C)\psi^I_{\alpha C}. \tag{2.59}$$

So the $\mathcal{N}=1$ supersymmetry does *not* commute with the $Sp(4)$ global symmetry. Since the matrix $\gamma_A{}^B$ contains four independent real parameters, equation (2.59) suggests that there are other 4 independent $\mathcal{N}=1$ supersymmetries. Therefore one may promote the $\mathcal{N}=1$ supersymmetry (2.57) to $\mathcal{N}=5$:

$$\delta Z^I_A = i\epsilon_A{}^{B\alpha} \psi^I_{B\alpha}, \tag{2.60}$$

where the parameter $\epsilon_A{}^{B\alpha} = \epsilon^\alpha_m \gamma^{mB}_A$. One may apply the same argument to the supersymmetry transformations of the fermionic and gauge fields. In summary, we have the following supersymmetry transformations:

$$\begin{aligned}
\delta Z^I_A &= i\epsilon_A{}^{B\alpha}\psi^I_{B\alpha},\\
\delta\psi^I_{A\alpha} &= (\gamma^\mu)_\alpha{}^\beta D_\mu Z^I_B \epsilon^B{}_{A\beta} + \frac{1}{3} f^I{}_{JKL}\omega^{BC} Z^J_B Z^K_C Z^L_D \epsilon^D{}_{A\alpha} - \frac{2}{3} f^I{}_{JKL}\omega^{BD} Z^J_C Z^K_D Z^L_A \epsilon^C{}_{B\alpha},\\
\delta \tilde{A}_\mu{}^K{}_L &= i\epsilon^{AB\alpha}(\gamma_\mu)_\alpha{}^\beta \psi^J_{B\beta} Z^I_A f_{IJ}{}^K{}_L,
\end{aligned} \tag{2.61}$$

where the parameter ϵ^{AB} is antisymmetric in AB, satisfying

$$\begin{aligned}
&\omega_{AB}\epsilon^{AB} = 0,\\
&\epsilon^*_{AB} = \omega^{AC}\omega^{BD}\epsilon_{CD}.
\end{aligned} \tag{2.62}$$

The supersymetry transformations are precisely the ones proposed in Ref. [38]. To verify the mechanism for enhancing the $\mathcal{N}=1$ to $\mathcal{N}=5$, it is best to check the closure of (2.61). Fortunately, the closure of (2.61) has been checked explicitly in Ref. [38]: they are indeed closed on-shell, and the corresponding equations of motion can be derived from the Lagrangian (2.54). So the R-symmetry of the theories is $Sp(4)$.

3. The $\mathcal{N}=4$ Theories and Symplectic Three-Algebras

3.1 $\mathcal{N}=4$ Theories by Starting from $\mathcal{N}=5$ Theories

In this section, we will construct the $\mathcal{N}$=4 theories by decomposing the $\mathcal{N}=5$ supermultiplets and the symplectic 3-algebra properly and proposing a new superpotential term that

preserving only $\mathcal{N}=4$. Let us first decompose the $\mathcal{N}=5$ super-fields for matter fields into $\mathcal{N}=4$ super-fields:

$$(\Phi^I_A)_{\mathcal{N}=5} = \begin{pmatrix} \Phi^a_A \\ \Phi^{a'}_{\dot{A}} \end{pmatrix} = \begin{pmatrix} Z^a_A \\ Z^{a'}_{\dot{A}} \end{pmatrix} + i \begin{pmatrix} 0 & \sigma_A{}^{\dot{A}} \\ \sigma^{\dagger}_{\dot{A}}{}^{A} & 0 \end{pmatrix} \begin{pmatrix} \psi^{a'}_{\dot{A}} \\ \psi^{a}_{\dot{A}} \end{pmatrix} - \frac{i}{2}\theta^2 \begin{pmatrix} F^a_A \\ F^{a'}_{\dot{A}} \end{pmatrix}. \tag{3.1}$$

The index A of the LHS runs from 1 to 4, while A and $\dot{A}$ of the RHS run from 1 to 2. (For the dotted and un-dotted representation, see Appendix A.3.) The indices a and a' run from 1 to $2M$ and 1 to $2N$, respectively. The superfields $\Phi^a_{\dot{A}}$ and $\Phi^{a'}_{\dot{A}}$ are called untwisted and twisted hyper-multiplets, repsectively, in the literature [33] (from the $\mathcal{N}=4$ point of view). The two antisymmetric matrices ω^{IJ} and ω^{AB} are decomposed as

$$\omega^{IJ} = \begin{pmatrix} \omega^{ab} & 0 \\ 0 & \omega^{a'b'} \end{pmatrix} \quad \text{and} \quad \omega^{AB} = \begin{pmatrix} \epsilon^{AB} & 0 \\ 0 & \epsilon^{\dot{A}\dot{B}} \end{pmatrix} \tag{3.2}$$

respectively. Now the reality condition $(\bar{\Phi}^A_I)_{\mathcal{N}=5} = \omega^{AB}\omega_{IJ}\Phi^J_B$ becomes

$$\bar{\Phi}^A_a = \epsilon^{AB}\omega_{ab}\Phi^b_B \quad \text{and} \quad \bar{\Phi}^{\dot{A}}_{a'} = \epsilon^{\dot{A}\dot{B}}\omega_{a'b'}\Phi^{b'}_{\dot{B}}. \tag{3.3}$$

To be compatible with the decomposition of the $\mathcal{N}=5$ hype-multiplets (3.1), one may decompose the $\mathcal{N}=5$ super-connections as

$$\Gamma^{IJ} f_{IJ}{}^K{}_L = \begin{pmatrix} \Gamma^{ab} f_{ab}{}^c{}_d + \Gamma^{a'b'} f_{a'b'}{}^c{}_d & 0 \\ 0 & \Gamma^{a'b'} f_{a'b'}{}^{c'}{}_{d'} + \Gamma^{ab} f_{ab}{}^{c'}{}_{d'} \end{pmatrix}, \tag{3.4}$$

where

$$\Gamma^{ab} f_{ab}{}^c{}_d = (i\theta^\beta A^{ab}_{\alpha\beta} + \theta^2\chi^{ab}_\alpha) f_{ab}{}^c{}_d, \tag{3.5}$$

and the other 3 superfields of the RHS of (3.4) have similar expressions. In proposing (3.4), we have decomposed the set of 3-algebra generators T_I into two sets of generators T_a and $T_{a'}$, and decomposed the 3-bracket (2.1) into 4 sets, with the structure constants $f_{abc}{}^d, f_{abc'}{}^{d'}, f_{a'b'c}{}^d$ and $f_{a'b'c'}{}^{d'}$. We have also decomposed the parameter superfield Γ^{IJ} into two superfields Γ^{ab} and $\Gamma^{a'b'}$.

If we introduce a 'spin up' spinor $\chi_{1\alpha}$ and a 'spin down' spinor $\chi_{2\alpha}$, i.e., [5]

$$\chi_{1\alpha} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \delta_{1\alpha} \quad \text{and} \quad \chi_{2\alpha} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \delta_{2\alpha}, \tag{3.6}$$

then in component formalism, we now have

$$f_{IJKL} = f_{abcd}\delta_{1\alpha}\delta_{1\beta}\delta_{1\gamma}\delta_{1\delta} + f_{abc'd'}\delta_{1\alpha}\delta_{1\beta}\delta_{2\gamma}\delta_{2\delta} + f_{a'b'cd}\delta_{2\alpha}\delta_{2\beta}\delta_{1\gamma}\delta_{1\delta} + f_{a'b'c'd'}\delta_{2\alpha}\delta_{2\beta}\delta_{2\gamma}\delta_{2\delta}, \tag{3.7}$$

(Here we assume that $(f_{abcd} - f_{abc'd'})$ does not vanish identically.) and

$$\Gamma^{IJ} = \Gamma^{ab}\delta_{1\alpha}\delta_{1\beta} + \Gamma^{a'b'}\delta_{2\alpha}\delta_{2\beta}. \tag{3.8}$$

[5] Here the index α is *not* an index of a spacetime spinor. We hope this will not cause any confusion.

Substituting (3.7) and (3.8) into $\Gamma^{IJ} f_{IJ}{}^K{}_L$ indeed gives (3.4). With the decomposition (3.7), the FI (2.5) are decomposed into 4 sets:

$$
\begin{aligned}
&f_{abe}{}^g f_{gfcd} + f_{abf}{}^g f_{egcd} - f_{efd}{}^g f_{abcg} - f_{efc}{}^g f_{abdg} = 0, \\
&f_{abe}{}^g f_{gfc'd'} + f_{abf}{}^g f_{egc'd'} - f_{efd'}{}^{g'} f_{abc'g'} - f_{efc'}{}^{g'} f_{abd'g'} = 0, \\
&f_{a'b'e}{}^g f_{gfc'd'} + f_{a'b'f}{}^g f_{egc'd'} - f_{efd'}{}^{g'} f_{a'b'c'g'} - f_{efc'}{}^{g'} f_{a'b'd'g'} = 0, \\
&f_{a'b'e'}{}^{g'} f_{g'f'c'd'} + f_{a'b'f'}{}^{g'} f_{e'g'c'd'} - f_{e'f'd'}{}^{g'} f_{a'b'c'g'} - f_{e'f'c'}{}^{g'} f_{a'b'd'g'} = 0.
\end{aligned}
\tag{3.9}
$$

In accordance with Eq. (2.16), these structure constants enjoy the symmetry properties

$$
\begin{aligned}
&f_{abcd} = f_{bacd} = f_{badc} = f_{cdab}, \\
&f_{abc'd'} = f_{bac'd'} = f_{bad'c'} = f_{c'd'ab}, \\
&f_{a'b'c'd'} = f_{b'a'c'd'} = f_{b'a'd'c'} = f_{c'd'a'b'}.
\end{aligned}
\tag{3.10}
$$

The reality condition (2.14) is decomposed into

$$
f^{*a}{}_b{}^c{}_d = f^b{}_a{}^d{}_c, \quad f^{*a'}{}_{b'}{}^c{}_d = f^{b'}{}_{a'}{}^d{}_c, \quad f^{*a'}{}_{b'}{}^{c'}{}_{d'} = f^{b'}{}_{a'}{}^{d'}{}_{c'}. \tag{3.11}
$$

Under the condition that $(f_{abcd} - f_{abc'd'})$ does not vanish identically, decomposing the constraint condition $f_{(IJK)L} = 0$ results in $f_{(abc)d} = 0$, $f_{(a'b'c')d'} = 0$ and $f_{abc'd'} = 0$. However, the condition $f_{abc'd'} = 0$ turns out to be too restrictive to allow any interaction between the primed fields and the un-primed fields. So we have to give up the constraint $f_{abc'd'} = 0$. Namely, we have to give up the constraint condition $f_{(IJK)L} = 0$ as we decompose f_{IJKL} by Eq. (3.7). Later we will see, to construct an interesting $\mathcal{N} = 4$ quiver gauge theory, we need only to impose constraints on f_{abcd} and $f_{a'b'c'd'}$:

$$
f_{(abc)d} = 0 \quad \text{and} \quad f_{(a'b'c')d'} = 0, \tag{3.12}
$$

while $f_{abc'd'}$ are un-constrained.

With these decompositions, the Lagrangian for the kinetic terms of the matter fields (2.32) becomes

$$
\begin{aligned}
\mathcal{L}_{\text{kin}} = &\frac{1}{2}(-D_\mu \bar{Z}^A_a D^\mu Z^a_A + i\bar{\psi}^{\dot{A}}_a \gamma^\mu D_\mu \psi^a_{\dot{A}} - 2i\sigma^\dagger_{\dot{B}}{}^A \bar{\psi}^{\dot{B}}_a \tilde{\chi}^a{}_b Z^b_A + \bar{F}^A_a F^a_A) \\
&+\frac{1}{2}(-D_\mu \bar{Z}^{\dot{A}}_{a'} D^\mu Z^{a'}_{\dot{A}} + i\bar{\psi}^A_{a'} \gamma^\mu D_\mu \psi^{a'}_A - 2i\sigma_B{}^{\dot{A}} \bar{\psi}^B_{a'} \tilde{\chi}^{a'}{}_{b'} Z^{b'}_{\dot{A}} + \bar{F}^{\dot{A}}_{a'} F^{a'}_{\dot{A}}),
\end{aligned}
\tag{3.13}
$$

where

$$
\begin{aligned}
D_\mu Z^A_d &= \partial_\mu Z^A_d - \tilde{A}_\mu{}^c{}_d Z^A_c, \\
\tilde{A}_\mu{}^c{}_d &= A^{ab}_\mu f_{ab}{}^c{}_d + A^{a'b'}_\mu f_{a'b'}{}^c{}_d, \\
\tilde{\chi}^{a'}{}_{b'} &= \chi^{cd} f_{cd}{}^{a'}{}_{b'} + \chi^{c'd'} f_{c'd'}{}^{a'}{}_{b'},
\end{aligned}
\tag{3.14}
$$

and similar definitions for $\tilde{A}_\mu{}^{c'}{}_{d'}$ and $\tilde{\chi}^a{}_b$; and the Chern-Simons term (2.35) becomes

$$\begin{aligned}\mathcal{L}_{\mathrm{CS}} = & \frac{1}{2}\epsilon^{\mu\nu\lambda}(f_{abcd}A^{ab}_\mu \partial_\nu A^{cd}_\lambda + \frac{2}{3}f_{abc}{}^g f_{gdef}A^{ab}_\mu A^{cd}_\nu A^{ef}_\lambda) \\ & +\frac{1}{2}\epsilon^{\mu\nu\lambda}(f_{a'b'c'd'}A^{a'b'}_\mu \partial_\nu A^{c'd'}_\lambda + \frac{2}{3}f_{a'b'c'}{}^{g'} f_{g'd'e'f'}A^{a'b'}_\mu A^{c'd'}_\nu A^{e'f'}_\lambda) \\ & +\epsilon^{\mu\nu\lambda}(f_{abc'd'}A^{ab}_\mu \partial_\nu A^{c'd'}_\lambda + f_{abc}{}^g f_{gde'f'}A^{ab}_\mu A^{cd}_\nu A^{e'f'}_\lambda + f_{abc'}{}^{g'} f_{g'd'e'f'}A^{ab}_\mu A^{c'd'}_\nu A^{e'f'}_\lambda) \\ & +\frac{i}{2}(f_{abcd}\chi^{ab}\chi^{cd} + 2f_{abc'd'}\chi^{ab}\chi^{c'd'} + f_{a'b'c'd'}\chi^{a'b'}\chi^{c'd'}).\end{aligned} \tag{3.15}$$

The equations of motion for the auxiliary field χ (2.36) is decomposed into two sets

$$\begin{aligned}\chi^{ab} &= -\sigma^{\dagger}_{\dot{A}}{}^{B}\psi^{\dot{A}(a}Z^{b)}_B, \\ \chi^{a'b'} &= -\sigma_A{}^{\dot{B}}\psi^{A(a'}Z^{b')}_{\dot{B}}.\end{aligned} \tag{3.16}$$

Plugging (3.16) into (3.13) and (3.15) gives three Yukawa terms

$$\begin{aligned}-\frac{i}{2}(&f_{acbd}\sigma^{A\dot{C}}\sigma^{B\dot{D}}Z^a_A Z^b_B \psi^c_{\dot{C}}\psi^d_{\dot{D}} + f_{a'c'b'd'}\sigma^{\dagger\dot{A}C}\sigma^{\dagger\dot{B}D}Z^{a'}_{\dot{A}}Z^{b'}_{\dot{B}}\psi^{c'}_C\psi^{d'}_D \\ &+2f_{abc'd'}\sigma^{A\dot{B}}\sigma^{\dagger\dot{C}D}Z^a_A Z^{c'}_{\dot{C}}\psi^b_{\dot{B}}\psi^{d'}_D).\end{aligned} \tag{3.17}$$

Alternatively, we can also obtain (3.17) by directly decomposing the $\mathcal{N}=5$ Yukawa term (2.37). It can be seen that the last term of (3.17) is a mixed term, in which the primed fields couple the un-primed fields through $f_{abc'd'}$. So we cannot obtain a non-trivial $\mathcal{N}=4$ superpotential by decomposing the $\mathcal{N}=5$ superpotential (2.40), because the $\mathcal{N}=5$ superpotential (2.40) is desired only if $f_{(IJK)L}=0$, which implies that $f_{abc'd'}=0$ as we decompose f_{IJKL} by Eq. (3.7) under the condition that $(f_{abcd}-f_{abc'd'})$ does not vanish identically. So we have to propose a new superpotential for the $\mathcal{N}=4$ theory, allowing $f_{abc'd'}\neq 0$. However, unlike the last term of (3.17), the first two terms of (3.17) are un-mixed terms. This inspires us to decompose the first term of the $\mathcal{N}=5$ superpotential (2.39) with $f_{a'c'bd}$ and $f_{acb'd'}$ deleted from f_{IKJL} (hence we denote the 'modified' structure constants as f'_{IKJL}):

$$\begin{aligned}W_1(\Phi) &= \frac{1}{12}(f'_{IKJL}\omega^{AB}\omega^{CD}\Phi^I_A\Phi^J_B\Phi^K_C\Phi^L_D)_{\mathcal{N}=5} \\ &= \frac{1}{12}(f_{acbd}\epsilon^{AB}\epsilon^{CD}\Phi^a_A\Phi^b_B\Phi^c_C\Phi^d_D + f_{a'c'b'd'}\epsilon^{\dot{A}\dot{B}}\epsilon^{\dot{C}\dot{D}}\Phi^{a'}_{\dot{A}}\Phi^{b'}_{\dot{B}}\Phi^{c'}_{\dot{C}}\Phi^{d'}_{\dot{D}}).\end{aligned} \tag{3.18}$$

where

$$f'_{IJKL} = f_{abcd}\delta_{1\alpha}\delta_{1\beta}\delta_{1\gamma}\delta_{1\delta} + f_{a'b'c'd'}\delta_{2\alpha}\delta_{2\beta}\delta_{2\gamma}\delta_{2\delta}. \tag{3.19}$$

Of course, the 'modified' structure constants f'_{IKJL} still satisfy the constraint condition $f'_{(IKJ)L}$=0, which is equivalent to Eq. (3.12): $f_{(acb)d}=0$ and $f_{(a'c'b')d'}=0$. We will prove that the first two terms of (3.17) combining the Yukawa terms arising from the superpotential W_1 (see (3.20)) are $SU(2)\times SU(2)$ invariant. Carrying out the Berezin

integral $\frac{i}{2}\int d\theta^2 W_1(\Phi)$ gives

$$\begin{aligned}\mathcal{L}_{W_1} = &-\frac{i}{6}(f_{acbd}\epsilon^{AB}\epsilon^{\dot{C}\dot{D}}Z^a_A Z^b_B \psi^c_{\dot{C}}\psi^d_{\dot{D}} + f_{a'c'b'd'}\epsilon^{\dot{A}\dot{B}}\epsilon^{CD}Z^{a'}_{\dot{A}}Z^{b'}_{\dot{B}}\psi^{c'}_C\psi^{d'}_D) \\ &-\frac{i}{6}[(f_{abcd}-f_{adcb})\sigma^{A\dot{C}}\sigma^{B\dot{D}}Z^a_A Z^b_B \psi^c_{\dot{C}}\psi^d_{\dot{D}} + (f_{a'b'c'd'}-f_{a'd'c'b'})\sigma^{\dagger\dot{A}C}\sigma^{\dagger\dot{B}D}Z^{a'}_{\dot{A}}Z^{b'}_{\dot{B}}\psi^{c'}_C\psi^{d'}_D] \\ &-\frac{1}{3}(f_{abcd}Z^b_B Z^{Bc}Z^{Ad}F^a_A + f_{a'b'c'd'}Z^{b'}_{\dot{B}}Z^{\dot{B}c'}Z^{\dot{A}d'}F^{a'}_{\dot{A}}).\end{aligned} \tag{3.20}$$

Let us now combine the first term of (3.17) and the first term of the second line of (3.20):

$$\begin{aligned}&-\frac{i}{6}[3f_{acbd} + (f_{abcd}-f_{adcb})]\sigma^{A\dot{C}}\sigma^{B\dot{D}}Z^a_A Z^b_B \psi^c_{\dot{C}}\psi^d_{\dot{D}} \\ &= -\frac{i}{6}(f_{acbd}-f_{bcad})(\sigma^{A\dot{C}}\sigma^{B\dot{D}} - \sigma^{B\dot{C}}\sigma^{A\dot{D}})Z^a_A Z^b_B \psi^c_{\dot{C}}\psi^d_{\dot{D}} \\ &= -\frac{i}{3}f_{acbd}\epsilon^{AB}\epsilon^{\dot{C}\dot{D}}Z^a_A Z^b_B \psi^c_{\dot{C}}\psi^d_{\dot{D}}.\end{aligned} \tag{3.21}$$

In the second line we have used $f_{(abc)d} = 0$. In the third line we have used the $SU(2)\times SU(2)$ identity (A.26). It can be seen that the final expression of (3.21) is indeed $SU(2)\times SU(2)$ invariant. Similarly, one can combine the second term of (3.17) and the second term of the second line of (3.20) to form an $SU(2)\times SU(2)$ invariant expression:

$$-\frac{i}{3}f_{a'c'b'd'}\epsilon^{\dot{A}\dot{B}}\epsilon^{CD}Z^{a'}_{\dot{A}}Z^{b'}_{\dot{B}}\psi^{c'}_C\psi^{d'}_D, \tag{3.22}$$

where we have used the reality condition (A.23). Now only the last term of (3.17), i.e. the mixed term, is not $SU(2)\times SU(2)$ invariant. Its structure suggests that if a Yukawa term of the form

$$if_{abc'd'}\sigma^{D\dot{B}}\sigma^{\dagger\dot{C}A}Z^a_A Z^{c'}_{\dot{C}}\psi^b_{\dot{B}}\psi^{d'}_D \tag{3.23}$$

arises from a to-be-determined superpotential, then they will add up to be $SU(2)\times SU(2)$ invariant by the reality condition (A.23) and the identity (A.26). It is therefore natural to try

$$W_2(\Phi) = \alpha f_{abc'd'}\sigma^{B\dot{D}}\sigma^{\dagger\dot{C}A}\Phi^a_A\Phi^b_B\Phi^{c'}_{\dot{C}}\Phi^{d'}_{\dot{D}}, \tag{3.24}$$

where α is a constant, to be determined later. The corresponding Lagrangian is

$$\begin{aligned}\mathcal{L}_{W_2} = &\, i\alpha f_{abc'd'}(\epsilon^{AC}\epsilon^{BD}Z^a_A Z^b_B \psi^{c'}_C\psi^{d'}_D + \epsilon^{\dot{A}\dot{C}}\epsilon^{\dot{B}\dot{D}}\psi^a_{\dot{A}}\psi^b_{\dot{B}}Z^{c'}_{\dot{C}}Z^{d'}_{\dot{D}} + 2\epsilon^{AC}\epsilon^{\dot{B}\dot{D}}Z^a_A Z^{d'}_{\dot{D}}\psi^b_{\dot{B}}\psi^{c'}_C) \\ &+2i\alpha f_{abc'd'}\sigma^{D\dot{B}}\sigma^{\dagger\dot{C}A}Z^a_A Z^{c'}_{\dot{C}}\psi^b_{\dot{B}}\psi^{d'}_D \\ &-2\alpha f_{abc'd'}\sigma^{B\dot{D}}\sigma^{\dagger\dot{C}A}Z^b_B Z^{c'}_{\dot{C}}Z^{d'}_{\dot{D}}F^a_A - 2\alpha f_{abc'd'}\sigma^{B\dot{D}}\sigma^{\dagger\dot{C}A}Z^a_A Z^b_B Z^{d'}_{\dot{D}}F^{c'}_{\dot{C}}.\end{aligned} \tag{3.25}$$

Note that the first line is $SU(2)\times SU(2)$ invariant by itself. Comparing the second line with (3.23) gives $\alpha = \frac{1}{2}$. Combining the last term of (3.17) and the second line of (3.25), we obtain

$$if_{abc'd'}\epsilon^{AD}\epsilon^{\dot{B}\dot{C}}Z^a_A Z^{c'}_{\dot{C}}\psi^b_{\dot{B}}\psi^{d'}_D, \tag{3.26}$$

which is the desired result. Now all Yukawa terms are invariant under the $SU(2)\times SU(2)$ global symmetry transformation. Put all Yukawa terms (the first line of (3.20), (3.21),

(3.22), (3.26) and the first line of (3.25)) together:

$$\mathcal{L}_{\mathrm{Y}} = -\frac{i}{2}(f_{acbd} Z^a_A Z^{Ab} \psi^c_{\dot{B}} \psi^{\dot{B}d} + f_{a'c'b'd'} Z^{a'}_{\dot{A}} Z^{\dot{A}b'} \psi^{c'}_B \psi^{Bd'}) \\ +\frac{i}{2} f_{abc'd'} (Z^a_A Z^b_B \psi^{Ac'} \psi^{Bd'} + Z^{c'}_{\dot{A}} Z^{d'}_{\dot{B}} \psi^{\dot{A}a} \psi^{\dot{B}b} + 4 Z^a_A Z^{\dot{B}d'} \psi^b_{\dot{B}} \psi^{Ac'}).$$

To calculate the bosonic potential, we first integrate out the auxiliary fields F^a_A and $F^{a'}_{\dot{A}}$ from (3.13), (3.20) and (3.25):

$$\bar{F}^A_a = \frac{1}{3} f_{abcd} Z^b_B Z^{Bc} Z^{Ad} + f_{abc'd'} \sigma^{B\dot{D}} \sigma^{\dagger \dot{C} A} Z^b_B Z^{c'}_{\dot{C}} Z^{d'}_{\dot{D}} \equiv W^A_{1a} + W^A_{2a}, \\ \bar{F}^{\dot{A}}_{a'} = \frac{1}{3} f_{a'b'c'd'} Z^{b'}_{\dot{B}} Z^{\dot{B}c'} Z^{\dot{A}d'} + f_{a'b'cd} \sigma^{\dagger \dot{B} D} \sigma^{C\dot{A}} Z^{b'}_{\dot{B}} Z^c_C Z^d_D \equiv W^{\dot{A}}_{1a'} + W^{\dot{A}}_{2a'}. \quad (3.27)$$

The bosonic potential is

$$-V = -\frac{1}{2}(\bar{F}^A_a F^a_A + \bar{F}^{\dot{A}}_{a'} F^{a'}_{\dot{A}}), \quad (3.28)$$

which is not manifestly $SU(2) \times SU(2)$ invariant due to the presence of the sigma matrices. However, by using the fundamental identities (3.9) and a method first introduced in GW theory [31] (see also [32]), we are able to re-write (3.28) so that it has a manifest $SU(2) \times SU(2)$ global symmetry. For example, let us consider

$$-W^A_{1a} W^a_{2A} = -\frac{1}{3} f_{abcd} f^a{}_{ec'd'} \sigma^{A\dot{C}} \sigma^{C\dot{D}} Z^b_B Z^{Bc} Z^d_A Z^e_C Z^{c'}_{\dot{C}} Z^{d'}_{\dot{D}} \\ = -\frac{1}{3}\{ f_{cda(b} f^a{}_{e)c'd'} + f_{cda[b} f^a{}_{e]c'd'} \} \sigma^{A\dot{C}} \sigma^{C\dot{D}} Z^b_B Z^{Bc} Z^d_A Z^e_C Z^{c'}_{\dot{C}} Z^{d'}_{\dot{D}} \\ \equiv S + A. \quad (3.29)$$

The antisymmetric part can be written as

$$A = \frac{1}{6} f_{cdab} f^a{}_{ec'd'} \sigma^{A\dot{C}} \sigma^{C\dot{D}} Z^{Bb} Z^e_B Z^c_C Z^d_A Z^{c'}_{\dot{C}} Z^{d'}_{\dot{D}}. \quad (3.30)$$

Applying the constraint condition $f_{(cdb)a} = 0$ to the above potential term, we obtain

$$A = -\frac{1}{3} f_{cdae} f^a{}_{bc'd'} \sigma^{A\dot{C}} \sigma^{C\dot{D}} Z^b_B Z^{Bc} Z^d_A Z^e_C Z^{c'}_{\dot{C}} Z^{d'}_{\dot{D}}. \quad (3.31)$$

Combining this with $-W^A_{1a} W^a_{2A}$ (the first line of (3.29)) gives

$$-W^A_{1a} W^a_{2A} + A = 2S. \quad (3.32)$$

Solving for $-W^A_{1a} W^a_{2A}$, we obtain

$$-W^A_{1a} W^a_{2A} = -\frac{1}{2} f_{cda(b} f^a{}_{e)c'd'} \sigma^{A\dot{C}} \sigma^{C\dot{D}} Z^b_B Z^{Bc} Z^d_A Z^e_C Z^{c'}_{\dot{C}} Z^{d'}_{\dot{D}}. \quad (3.33)$$

Let us now consider another term of (3.28):

$$-\frac{1}{2} W^{\dot{A}}_{2a'} W^{a'}_{2\dot{A}} = \frac{1}{2} f_{cdb'a'} f^{a'}{}_{e'fg} \sigma^{D\dot{B}} \sigma^{A\dot{F}} Z^{b'}_{\dot{B}} Z^{e'}_{\dot{F}} Z^c_C Z^d_D Z^{Cf} Z^g_A \quad (3.34) \\ = \frac{1}{2}(f_{cda'(b'} f^{a'}{}_{e')fg} + f_{cda'[b'} f^{a'}{}_{e']fg}) \sigma^{D\dot{B}} \sigma^{A\dot{F}} Z^{b'}_{\dot{B}} Z^{e'}_{\dot{F}} Z^c_C Z^d_D Z^{Cf} Z^g_A.$$

Combining this equation with (3.33), the symmetric part cancels (3.33) by the second equation of the fundamental identities (3.9), while the antisymmetric part is $SU(2)\times SU(2)$ invariant by the identity (A.26). The final result is

$$-W^A_{1a}W^a_{2A} - \frac{1}{2}W^{\dot{A}}_{2a'}W^{a'}_{2\dot{A}} = -\frac{1}{4}f_{abc'g'}f^{g'}{}_{d'ef}Z^{\dot{A}c'}Z^{d'}_{\dot{A}}Z^b_D Z^{Df}Z^a_C Z^{Ce}. \tag{3.35}$$

One can apply the same method to the other terms of (3.28). The final expression for the $\mathcal{N}=4$ bosonic potential is

$$\begin{aligned}-V = {}&+\frac{1}{12}(f_{abcg}f^g{}_{def}Z^{Aa}Z^b_B Z^{B(c}Z^{d)}_C Z^{Ce}Z^f_A + f_{a'b'c'g'}f^{g'}{}_{d'e'f'}Z^{\dot{A}a'}Z^{b'}_{\dot{B}}Z^{\dot{B}(c'}Z^{d')}_{\dot{C}}Z^{\dot{C}e'}Z^{f'}_{\dot{A}})\\ &-\frac{1}{4}(f_{abc'g'}f^{g'}{}_{d'ef}Z^{\dot{A}c'}Z^{d'}_{\dot{A}}Z^b_D Z^{Df}Z^a_C Z^{Ce} + f_{a'b'cg}f^g{}_{de'f'}Z^{Ac}Z^d_A Z^{b'}_{\dot{D}}Z^{\dot{D}f'}Z^{a'}_{\dot{C}}Z^{\dot{C}e'})\end{aligned} \tag{3.36}$$

In summary, the full $\mathcal{N}=4$ Lagrangian is given by

$$\begin{aligned}\mathcal{L} = {}&\frac{1}{2}(-D_\mu\bar{Z}^A_a D^\mu Z^a_A - D_\mu\bar{Z}^{\dot{A}}_{a'}D^\mu Z^{a'}_{\dot{A}} + i\bar{\psi}^{\dot{A}}_a\gamma^\mu D_\mu\psi^a_{\dot{A}} + i\bar{\psi}^A_{a'}\gamma^\mu D_\mu\psi^{a'}_A)\\ &-\frac{i}{2}(f_{acbd}Z^a_A Z^{Ab}\psi^c_{\dot{B}}\psi^{\dot{B}d} + f_{a'c'b'd'}Z^{a'}_{\dot{A}}Z^{\dot{A}b'}\psi^{c'}_B\psi^{Bd'})\\ &+\frac{i}{2}f_{abc'd'}(Z^a_A Z^b_B\psi^{Ac'}\psi^{Bd'} + Z^{c'}_{\dot{A}}Z^{d'}_{\dot{B}}\psi^{\dot{A}a}\psi^{\dot{B}b} + 4Z^a_A Z^{\dot{B}d'}\psi^b_{\dot{B}}\psi^{Ac'})\\ &+\frac{1}{2}\epsilon^{\mu\nu\lambda}(f_{abcd}A^{ab}_\mu\partial_\nu A^{cd}_\lambda + \frac{2}{3}f_{abc}{}^g f_{gdef}A^{ab}_\mu A^{cd}_\nu A^{ef}_\lambda)\\ &+\frac{1}{2}\epsilon^{\mu\nu\lambda}(f_{a'b'c'd'}A^{a'b'}_\mu\partial_\nu A^{c'd'}_\lambda + \frac{2}{3}f_{a'b'c'}{}^{g'}f_{g'd'e'f'}A^{a'b'}_\mu A^{c'd'}_\nu A^{e'f'}_\lambda)\\ &+\epsilon^{\mu\nu\lambda}(f_{abc'd'}A^{ab}_\mu\partial_\nu A^{c'd'}_\lambda + f_{abc}{}^g f_{gde'f'}A^{ab}_\mu A^{cd}_\nu A^{e'f'}_\lambda + f_{abc'}{}^{g'}f_{g'd'e'f'}A^{ab}_\mu A^{c'd'}_\nu A^{e'f'}_\lambda)\\ &+\frac{1}{12}(f_{abcg}f^g{}_{def}Z^{Aa}Z^b_B Z^{B(c}Z^{d)}_C Z^{Ce}Z^f_A + f_{a'b'c'g'}f^{g'}{}_{d'e'f'}Z^{\dot{A}a'}Z^{b'}_{\dot{B}}Z^{\dot{B}(c'}Z^{d')}_{\dot{C}}Z^{\dot{C}e'}Z^{f'}_{\dot{A}})\\ &-\frac{1}{4}(f_{abc'g'}f^{g'}{}_{d'ef}Z^{\dot{A}c'}Z^{d'}_{\dot{A}}Z^b_D Z^{Df}Z^a_C Z^{Ce} + f_{a'b'cg}f^g{}_{de'f'}Z^{Ac}Z^d_A Z^{b'}_{\dot{D}}Z^{\dot{D}f'}Z^{a'}_{\dot{C}}Z^{\dot{C}e'}).\end{aligned} \tag{3.37}$$

Using the same argument given in Sec. 2.2, we may promote the $\mathcal{N}=1$ supersymmetry transformations to $\mathcal{N}=4$:

$$\begin{aligned}\delta Z^a_A &= i\epsilon_A{}^{\dot{A}}\psi^a_{\dot{A}},\\ \delta Z^{a'}_{\dot{A}} &= i\epsilon^\dagger_{\dot{A}}{}^A\psi^{a'}_A,\\ \delta\psi^{a'}_A &= -\gamma^\mu D_\mu Z^{a'}_{\dot{B}}\epsilon_A{}^{\dot{B}} - \frac{1}{3}f^{a'}{}_{b'c'd'}Z^{b'}_{\dot{B}}Z^{\dot{B}c'}Z^{d'}_{\dot{C}}\epsilon_A{}^{\dot{C}} + f^{a'}{}_{b'cd}Z^{b'}_{\dot{A}}Z^{Bc}Z^d_A\epsilon_B{}^{\dot{A}},\\ \delta\psi^a_{\dot{A}} &= -\gamma^\mu D_\mu Z^a_B\epsilon^\dagger_{\dot{A}}{}^B - \frac{1}{3}f^a{}_{bcd}Z^b_B Z^{Bc}Z^d_C\epsilon^\dagger_{\dot{A}}{}^C + f^a{}_{bc'd'}Z^b_A Z^{\dot{B}c'}Z^{d'}_{\dot{A}}\epsilon^\dagger_{\dot{B}}{}^A,\\ \delta\tilde{A}_\mu{}^c{}_d &= i\epsilon^{A\dot{B}}\gamma_\mu\psi^b_{\dot{B}}Z^a_A f_{ab}{}^c{}_d + i\epsilon^{\dagger\dot{A}B}\gamma_\mu\psi^{b'}_B Z^{a'}_{\dot{A}}f_{a'b'}{}^c{}_d,\\ \delta\tilde{A}_\mu{}^{c'}{}_{d'} &= i\epsilon^{A\dot{B}}\gamma_\mu\psi^b_{\dot{B}}Z^a_A f_{ab}{}^{c'}{}_{d'} + i\epsilon^{\dagger\dot{A}B}\gamma_\mu\psi^{b'}_B Z^{a'}_{\dot{A}}f_{a'b'}{}^{c'}{}_{d'},\end{aligned} \tag{3.38}$$

where the parameter satisfies the reality condition

$$\epsilon^\dagger{}_{\dot{A}}{}^B = -\epsilon^{BC}\epsilon_{\dot{A}\dot{B}}\epsilon_C{}^{\dot{B}}. \tag{3.39}$$

It is still necessary to verify the closure of the $\mathcal{N}=4$ superalgebra; this will be done in the next subsection. The ordinary Lie algebra counterparts of the Lagrangian (3.37) and the supersymmetry transformaitons (3.38) are first constructed in Ref. [32]. If $f_{abc'd'}=f_{abcd}$, then $f_{abc'd'}$ also satisfy the constraint equation, i.e. $f_{(abc')d'}=0$. In this special case, the $\mathcal{N}=4$ supersymmetry will be enhanced to $\mathcal{N}=5$. Therefore without the twisted hypermultiplet, it is impossible to enhance the $\mathcal{N}=4$ supersymmetry to $\mathcal{N}=5$; as a result, the $\mathcal{N}=4$ supersymmetry cannot be promoted to $\mathcal{N}=6,8$. Indeed, in Ref. [32], it was demonstrated that the $\mathcal{N}=4$ theory with an $SU(2)\times SU(2)$ gauge group is equivalent to the $\mathcal{N}=8$ BLG theory *after* adding the twisted hypermultiplet.

In a forthcoming paper [44], we will convert the $\mathcal{N}=4$ theories (based on the 3-algebras) into general $\mathcal{N}=4$ theories in terms of ordinary Lie (2-)algebras, using superalgebras to realize the 3-algebras. The method will be generalized to construct $\mathcal{N}=4$ quiver gauge theories [44]. There are a special class of $\mathcal{N}=4$ theories, with a circular quiver gauge diagram [32, 44]:

$$\cdots - U(N_{i-1}) - U(N_i) - U(N_{i+1}) - \cdots . \tag{3.40}$$

(The above diagram is only a part of the circular quiver gauge diagram.) This class of $\mathcal{N}=4$ theories have been conjectured to be the gauge descriptions of multi M2-branes in orbifold $(\mathbf{C}^2/\mathbf{Z}_p\times\mathbf{C}^2/\mathbf{Z}_q)/\mathbf{Z}_k$, where p (q) is the number of the un-twisted (twisted) hypermultiplets, and k the Chern-Simons level [45]. Their gravity duals have been investigated in Ref. [45]. To our knowledge, most of the gravity duals of the $\mathcal{N}=4$ quiver gauge theories are not found yet. We would like to construct their gravity duals in the future.

If one sets the twisted hypermultiplet to be zero, i.e., $\Phi^{a'}_{\dot{A}}=0$, then (3.37) and (3.38) become the Lagrangian and the supersymmetry law of the GW theory [31], respectively, in the 3-algebra approach:

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}(-D_\mu \bar{Z}^A_a D^\mu Z^a_A + i\bar{\psi}^{\dot{A}}_a\gamma^\mu D_\mu\psi^a_{\dot{A}}) - \frac{i}{2}f_{acbd}Z^a_A Z^{Ab}\psi^c_{\dot{B}}\psi^{\dot{B}d}\\ & +\frac{1}{2}\epsilon^{\mu\nu\lambda}(f_{abcd}A^{ab}_\mu\partial_\nu A^{cd}_\lambda + \frac{2}{3}f_{abc}{}^g f_{gdef}A^{ab}_\mu A^{cd}_\nu A^{ef}_\lambda)\\ & +\frac{1}{12}f_{abcg}f^g{}_{def}Z^{Aa}Z^b_B Z^{B(c}Z^{d)}_C Z^{Ce}Z^f_A,\end{aligned} \tag{3.41}$$

and

$$\begin{aligned}\delta Z^a_A &= i\epsilon_A{}^{\dot{A}}\psi^a_{\dot{A}},\\ \delta\psi^a_{\dot{A}} &= -\gamma^\mu D_\mu Z^a_B\epsilon^{\dagger}_{\dot{A}}{}^B - \frac{1}{3}f^a{}_{bcd}Z^b_B Z^{Bc}Z^d_C\epsilon^{\dagger}_{\dot{A}}{}^C,\\ \delta\tilde{A}_\mu{}^c{}_d &= i\epsilon^{A\dot{B}}\gamma_\mu\psi^b_{\dot{B}}Z^a_A f_{ab}{}^c{}_d.\end{aligned} \tag{3.42}$$

3.2 Closure of the N=4 Algebra

The closure of the algebra of the GW theory was checked in [31]. To our knowledge, there is no explicit check in the literature for the closure of the $\mathcal{N}=4$ algebra after adding the

twisted hypermultiplets into the GW theory. Here we present such a check by starting with the supersymmetry transformation of the scalar fields:

$$[\delta_1, \delta_2] Z^a_A = v^\mu D_\mu Z^a_A + \frac{1}{3} f^a{}_{bcd} Z^b_B Z^c_C Z^d_D \epsilon_{AE} \epsilon^{BC} u^{ED}$$
$$+ i f^a{}_{bc'd'} Z^{Bb} Z^{\dot{B}c'} Z^{\dot{A}d'} (\epsilon_{2A\dot{A}} \epsilon^\dagger_{1\dot{B}B} - \epsilon_{1A\dot{A}} \epsilon^\dagger_{2\dot{B}B}), \tag{3.43}$$

where

$$v^\mu \equiv i \epsilon^\dagger_{1\dot{A}}{}^B \gamma^\mu \epsilon_{2B}{}^{\dot{A}}, \qquad u^{ED} \equiv i(\epsilon_1^{E\dot{A}} \epsilon^\dagger_{2\dot{A}}{}^D - \epsilon_2^{E\dot{A}} \epsilon^\dagger_{1\dot{A}}{}^D). \tag{3.44}$$

By using the identity $\epsilon_{AE}\epsilon^{BC} = -(\delta_A{}^B \delta_E{}^C - \delta_E{}^B \delta_A{}^C)$, the second term of the RHS of (3.43) can be written as

$$-\frac{1}{3} f^a{}_{bcd} Z^b_A Z^c_C Z^d_D u^{CD} + \frac{1}{3} f^a{}_{bcd} Z^b_B Z^c_A Z^d_D u^{BD}. \tag{3.45}$$

The second term is equal to the first term minus the second term by the constraint condition $f^a{}_{(bcd)} = 0$:

$$\frac{1}{3} f^a{}_{bcd} Z^b_B Z^c_A Z^d_D u^{BD} = -\frac{1}{3} f^a{}_{bcd} Z^b_A Z^c_C Z^d_D u^{CD} - \frac{1}{3} f^a{}_{bcd} Z^b_B Z^c_A Z^d_D u^{BD}. \tag{3.46}$$

Therefore the second term of the RHS of (3.43) is equal to

$$-\frac{1}{2} f^a{}_{bcd} Z^c_C Z^d_D u^{CD} Z^b_A. \tag{3.47}$$

By using the fourth equation of (A.27), the second line of the RHS of (3.43) becomes

$$-\frac{1}{2} f^a{}_{bc'd'} Z^{c'}_{\dot{A}} Z^{d'}_{\dot{B}} u^{\dot{A}\dot{B}} Z^b_A, \tag{3.48}$$

where

$$u^{\dot{A}\dot{B}} \equiv i(\epsilon_1^{\dagger\dot{A}C} \epsilon_{2C}{}^{\dot{B}} - \epsilon_2^{\dagger\dot{A}C} \epsilon_{1C}{}^{\dot{B}}). \tag{3.49}$$

In summary, we have

$$[\delta_1, \delta_2] Z^a_A = v^\mu D_\mu Z^a_A + \tilde{\Lambda}^a{}_b Z^b_A. \tag{3.50}$$

While the first is the familiar covariant derivative, the second term is a gauge transformation by a parameter

$$\tilde{\Lambda}^a{}_b \equiv -\frac{1}{2} f^a{}_{bcd} Z^c_C Z^d_D u^{CD} - \frac{1}{2} f^a{}_{bc'd'} Z^{c'}_{\dot{A}} Z^{d'}_{\dot{B}} u^{\dot{A}\dot{B}}. \tag{3.51}$$

Similarly, we have

$$[\delta_1, \delta_2] Z^{a'}_{\dot{A}} = v^\mu D_\mu Z^{a'}_{\dot{A}} + \tilde{\Lambda}^{a'}{}_{b'} Z^{b'}_{\dot{A}}, \tag{3.52}$$

where the parameter $\tilde{\Lambda}^{a'}{}_{b'}$ is defined as

$$\tilde{\Lambda}^{a'}{}_{b'} \equiv -\frac{1}{2} f^{a'}{}_{b'c'd'} Z^{c'}_{\dot{C}} Z^{d'}_{\dot{D}} u^{\dot{C}\dot{D}} - \frac{1}{2} f^{a'}{}_{b'cd} Z^c_A Z^d_B u^{AB}. \tag{3.53}$$

Let us now examine the supersymmetry transformation of the gauge fields:

$$\begin{aligned}
[\delta_1, \delta_2]\tilde{A}_\mu{}^a{}_b = & \; v^\nu \tilde{F}_{\nu\mu}{}^a{}_b - D_\mu \tilde{\Lambda}^a{}_b \\
& + v^\nu \{ \tilde{F}_{\mu\nu}{}^a{}_b - \varepsilon_{\mu\nu\lambda}[(Z^c_A D^\lambda \bar{Z}^{Ad} - \frac{i}{2}\bar{\psi}^{\dot{B}c}\gamma^\lambda \psi^d_{\dot{B}}) f_{cd}{}^a{}_b \\
& + (Z^{c'}_{\dot{A}} D^\lambda \bar{Z}^{\dot{A}d'} - \frac{i}{2}\bar{\psi}^{Bc'}\gamma^\lambda \psi^{d'}_B) f_{c'd'}{}^a{}_b] \} \\
& + \mathcal{O}(Z^4).
\end{aligned} \tag{3.54}$$

The last term $\mathcal{O}(Z^4)$, which is fourth order in the scalar fields Z, vanishes by the FI (3.9). The second line and the third line must be the equations of motion for the gauge fields:

$$\tilde{F}_{\mu\nu}{}^a{}_b = \varepsilon_{\mu\nu\lambda}[(Z^c_A D^\lambda \bar{Z}^{Ad} - \frac{i}{2}\bar{\psi}^{\dot{B}c}\gamma^\lambda \psi^d_{\dot{B}}) f_{cd}{}^a{}_b + (Z^{c'}_{\dot{A}} D^\lambda \bar{Z}^{\dot{A}d'} - \frac{i}{2}\bar{\psi}^{Bc'}\gamma^\lambda \psi^{d'}_B) f_{c'd'}{}^a{}_b], \tag{3.55}$$

while the first line remains:

$$[\delta_1, \delta_2]\tilde{A}_\mu{}^a{}_b = v^\nu \tilde{F}_{\nu\mu}{}^a{}_b - D_\mu \tilde{\Lambda}^a{}_b. \tag{3.56}$$

The first term is a covariant translation; the second term is a gauge transformation, as expected. Similarly, we have

$$[\delta_1, \delta_2]\tilde{A}_\mu{}^{a'}{}_{b'} = v^\nu \tilde{F}_{\nu\mu}{}^{a'}{}_{b'} - D_\mu \tilde{\Lambda}^{a'}{}_{b'}, \tag{3.57}$$

and

$$\tilde{F}_{\mu\nu}{}^{a'}{}_{b'} = \varepsilon_{\mu\nu\lambda}[(Z^{c'}_{\dot{A}} D^\lambda \bar{Z}^{\dot{A}d'} - \frac{i}{2}\bar{\psi}^{Bc'}\gamma^\lambda \psi^{d'}_B) f_{c'd'}{}^{a'}{}_{b'} + (Z^c_A D^\lambda \bar{Z}^{Ad} - \frac{i}{2}\bar{\psi}^{\dot{B}c}\gamma^\lambda \psi^d_{\dot{B}}) f_{cd}{}^{a'}{}_{b'}]. \tag{3.58}$$

Finally we examine the fermion supersymmetry transformation:

$$\begin{aligned}
[\delta_1, \delta_2]\psi^a_{\dot{A}} = & \; v^\mu D_\mu \psi^a_{\dot{A}} + \tilde{\Lambda}^a{}_b \psi^b_{\dot{A}} \\
& - \frac{i}{2}(\epsilon_1^{\dagger \dot{C}B}\epsilon_{2B\dot{A}} - \epsilon_2^{\dagger \dot{C}B}\epsilon_{1B\dot{A}}) E^a_{\dot{C}} \\
& - \frac{1}{2} v_\nu \gamma^\nu E^a_{\dot{A}},
\end{aligned} \tag{3.59}$$

where

$$E^a_{\dot{A}} = \gamma^\mu D_\mu \psi^a_{\dot{A}} + f_{cdb}{}^a Z^b_B Z^{Bc} \psi^d_{\dot{A}} - f_{c'd'b}{}^a Z^{c'}_{\dot{A}} Z^{d'}_{\dot{C}} \psi^{\dot{C}b} + 2 f_{c'd'b}{}^a Z^b_B Z^{c'}_{\dot{A}} \psi^{Bd'}. \tag{3.60}$$

In order to achieve the closure of the algebra, we must impose the equations of motion for the fermionic fields:

$$E^a_{\dot{A}} = 0. \tag{3.61}$$

As a result, only the first line of (3.59) remains. Similarly, we obtain

$$[\delta_1, \delta_2]\psi^{a'}_A = v^\mu D_\mu \psi^{a'}_A + \tilde{\Lambda}^{a'}{}_{b'} \psi^{b'}_A,$$

and

$$0 = E^{a'}_A = \gamma^\mu D_\mu \psi^{a'}_A + f_{c'd'b'}{}^{a'} Z^{b'}_{\dot{B}} Z^{\dot{B}c'} \psi^{d'}_A - f_{cdb'}{}^{a'} Z^c_A Z^d_C \psi^{Cb'} + 2 f_{cdb'}{}^{a'} Z^{b'}_{\dot{B}} Z^c_A \psi^{\dot{B}d}. \tag{3.62}$$

One can derive all the equations of motion of as the Euler-Lagrangian equations from the Lagrangian (3.37).

4. Three-algebras, Lie superalgebras and Embedding Tensors

4.1 Three-algebras and Lie superalgebras

In this section, we will demonstrate that the symplectic 3-algebra can be realized in terms of a super Lie algebra.

Recall that in Sec. 2.2, we note that f_{IJKL} can be specified as $k_{mn}\tau^m_{IJ}\tau^n_{KL}$ (up to an unimportant constant), i.e.

$$f_{IJKL} = k_{mn}\tau^m_{IJ}\tau^n_{KL}, \tag{4.1}$$

where the set of matrices τ^m_{IK} is in the fundamental representation of $Sp(2L)$ or its subalgebra, and k_{mn} is the Killing-Cartan metric.

Further more, the constraint condition $f_{(IJK)L} = 0$ implies that $f_{(IJK)L} = k_{mn}\tau^m_{(IJ}\tau^n_{K)L} = 0$. As GW pointed out [31], the constraint equation $k_{mn}\tau^m_{(IJ}\tau^n_{K)L} = 0$ can be solved in terms of the Jacobi identity for following super Lie algebra: [6]

$$\begin{aligned}
&[M^m, M^n] = C^{mn}{}_s M^s,\\
&[M^m, Q_I] = -\tau^m_{IJ}\omega^{JK}Q_K,\\
&\{Q_I, Q_J\} = \tau^m_{IJ}k_{mn}M^n.
\end{aligned} \tag{4.2}$$

Namely, the QQQ Jacobi identity

$$[\{Q_I, Q_J\}, Q_K] + [\{Q_J, Q_K\}, Q_I] + [\{Q_K, Q_I\}, Q_J] = 0 \tag{4.3}$$

is equivalent to the constraint equation $k_{mn}\tau^m_{(IJ}\tau^n_{K)L} = 0$. Therefore GW's approach suggests that the symplectic 3-algebra can be realized in terms of the super Lie algebra (4.2), if we think of the 3-algebra generator T_I as the fermionic generator Q_I. Comparing the 3-bracket $[T_I, T_J; T_K] = f_{IJK}{}^L T_l$ with

$$[\{Q_I, Q_J\}, Q_K] = k_{mn}\tau^m_{IJ}\tau^n_K{}^L Q_L, \tag{4.4}$$

and taking account of (4.1), we see that the 3-bracket may be realized in terms of the double graded commutator

$$[T_I, T_J; T_K] \doteq [\{Q_I, Q_J\}, Q_K]. \tag{4.5}$$

Here the RHS is also obviously symmetric in IJ. It is instructive to examine the FI (2.4) with the 3-brackets replaced by the double graded commutators:

$$\begin{aligned}
&[\{Q_I, Q_J\}, [\{Q_M, Q_N\}, Q_K]] \\
&= [\{[\{Q_I, Q_J\}, Q_M], Q_N\}, Q_K] + [\{Q_M, [\{Q_I, Q_J\}, Q_N]\}, Q_K] + [\{Q_M, Q_N\}, [\{Q_I, Q_J\}, Q_K]].
\end{aligned} \tag{4.6}$$

By using the super Lie algebra (4.2), we obtain

$$\tau^m_{IJ}\tau^n_{MN}([M_n, [M_m, Q_K]] - [M_m, [M_n, Q_K]] + [[M_m, M_n], Q_K]) = 0, \tag{4.7}$$

[6] This is *not* the $D = 3$ super-Pioncare algebra.

which is equivalent to the MMQ Jacobi identity of the super Lie algebra (4.2). It is not difficult to prove that $k_{mn}\tau^m_{IJ}\tau^n_{KL}$ also enjoy the symmetry properties (2.16). So indeed the symplectic 3-algebra can be realized in terms of the super Lie algebra. Now recall the component formulism of the basic definition of the global transformation

$$\delta_{\tilde{\Lambda}}X^K = \Lambda^{IJ}f_{IJ}{}^K{}_L X^L. \tag{4.8}$$

Replacing $f_{IJ}{}^K{}_L$ by $k_{mn}\tau^m_{IJ}\tau^{nK}{}_L$ gives

$$\delta_{\tilde{\Lambda}}X^K = \Lambda^{IJ}k_{mn}\tau^m_{IJ}\tau^{nK}{}_L X^L. \tag{4.9}$$

From the ordinary Lie group point of view, this is a transformation with parameters $\Lambda^{IJ}k_{mn}\tau^m_{IJ}$ and generators $\tau^{nK}{}_L$. On the other hand, the second equation of (4.2) indicates that the fermionic generators furnish a representation of the bosonic part of the super Lie algebra (4.2), i.e. the matrix τ^m_{IJ} is a quaternion representation of M^m. Therefore, the gauge group generated by the 3-algebra can be determined as follows: its Lie algebra is just the bosonic part of the super Lie algebra (4.2), which must be $Sp(2L)$ or its sub-algebras. The representation of the matter fields is determined by the fermionic generators of the super Lie algebra (4.2).

For a more mathematical approach, see Ref. [35, 40, 41], in which the relations between the 3-algebras and Lie superalgebras are discussed by using Lie algebra representation theories.

4.2 Three-algebras and Lie Algebras

It is less obvious that one can also prove that (4.1) is an explicit solution of the FI (2.5) by using the QQM Jacobi identity of the super Lie algebra, which reads

$$[\{Q_I, Q_J\}, M^m] - \{[Q_J, M^m], Q_I\} + \{[M^m, Q_I], Q_J\} = 0. \tag{4.10}$$

After a short algebra we obtain

$$\tau^n_{IJ}k_{np}[M^p, M^m] - \tau^{mK}{}_J\tau^n_{KI}k_{np}M^p - \tau^{mK}{}_I\tau^n_{KJ}k_{np}M^p = 0. \tag{4.11}$$

Since the matrix τ^m_{IJ} is a representation of M^m, the above equation implies

$$\tau^n_{IJ}k_{np}[\tau^p, \tau^m]_{MN} - \tau^{mK}{}_J\tau^n_{KI}k_{np}\tau^p_{MN} - \tau^{mK}{}_I\tau^n_{KJ}k_{np}\tau^p_{MN} = 0, \tag{4.12}$$

where

$$[\tau^p, \tau^m]_{MN} = \tau^p_{MO}\tau^{mO}{}_N - \tau^m_{MO}\tau^{pO}{}_N. \tag{4.13}$$

Multiplying both sides by $k_{mq}\tau^q_{KL}$ gives

$$k_{np}\tau^n_{IJ}k_{mq}\tau^q_{KL}[\tau^p, \tau^m]_{MN} - k_{mq}\tau^q_{KL}\tau^{mK}{}_J\tau^n_{KI}k_{np}\tau^p_{MN} - k_{mq}\tau^q_{KL}\tau^{mK}{}_I\tau^n_{KJ}k_{np}\tau^p_{MN} = 0. \tag{4.14}$$

Rearranging the above equation verifies explicitly that (4.1) satisfies the FI (2.5). Application of the commutator

$$[\tau^m, \tau^n]_{IJ} = C^{mn}{}_p\tau^p_{IJ} \tag{4.15}$$

to Eq. (4.14) gives

$$(k_{np}k_{qm}C^{pm}{}_s + k_{qm}k_{sp}C^{pm}{}_n)\tau^n_{IJ}\tau^q_{KL}\tau^s_{MN} = 0. \tag{4.16}$$

Here the equation in the bracket is simply the statement that the structure constants

$$\tilde{C}_{nqs} = k_{np}k_{qm}C^{pm}{}_s \tag{4.17}$$

are totally antisymmetric if the three adjoint indices nqs are on equal footing. Note that k_{mn} is an invariant bilinear form on the bosonic subalgebra, since Eq. (4.16) or (4.17) also implies

$$[k, C^m] = 0. \tag{4.18}$$

Here the matrices $(C^m)^p{}_n = C^{mp}{}_n$ furnish the usual adjoint representation of the bosonic subalgebra. In this way, we see that the FI of the 3-algebra can be converted into two ordinary commutators (4.15) and (4.18) (this is first discovered in the second paper of Ref. [15] with a different approach).

Eq. (4.9) indicates that $f_{IJKL} = k_{mn}\tau^m_{IJ}\tau^n_{KL}$ also furnish a quaternion representation of the bosonic subalgebra. In fact, if we write f_{IJKL} as $(f_{IJ})_{KL}$, then f_{IJ} is a set of matrices, and corresponding matrix elements are $(f_{IJ})_{KL}$. If τ^n_{KL} furnish a quaternion of representation of M^n, then $(f_{IJ})_{KL}$ furnish a quaternion representation of $M_{IJ} = k_{mn}\tau^m_{IJ}M^n$, since the operator M_{IJ} is a linear combination of M^n. With this understanding, we are able to re-write the FI (2.5) as a commutator

$$\begin{aligned}[]
[f_{IJ}, f_{KL}]_{MN} &= C_{IJ,KL}{}^{OP}(f_{OP})_{MN} \\
&= (f_{IJK}{}^{O}\delta^P_L + f_{IJL}{}^{O}\delta^P_K)(f_{OP})_{MN} \\
&= -[f_{IJ}, f_{MN}]_{KL}.
\end{aligned} \tag{4.19}$$

The third equation says that the quantity $[f_{IJ}, f_{KL}]_{MN}$ are totally antisymmetric in the 3 pairs of indices. Eq. (4.19) is equivalent to Eq. (4.15). Also, the matrices $(f_{IJ})_{KL}$ satisfy the conventional Jacobi identity as a result of the MMM Jacobi identity of the superalgebra of (4.2). We now must check whether $\tilde{C}_{IJ,KL,MN} = k_{MN,OP}C_{IJ,KL}{}^{OP}$ are totally antisymmetric or not. To be consistent with the transformation $M_{IJ} = k_{mn}\tau^m_{IJ}M^n$, we must transform the Killing-Cartan metric k^{mn} as

$$k^{mn} \to k_{IJ,KL} = k_{qm}\tau^q_{IJ}k_{pn}\tau^p_{KL}k^{mn} = k_{mn}\tau^m_{IJ}\tau^n_{KL} = f_{IJKL}. \tag{4.20}$$

Namely the structure constants f_{IJKL} also play a role of the Killing-Cartan metric $k_{IJ,KL}$. So we must use f_{MNOP} to lower the OP indices of $C_{IJ,KL}{}^{OP}$: [7]

$$\begin{aligned}
\tilde{C}_{IJ,KL,MN} &= f_{MNOP}C_{IJ,KL}{}^{OP} \\
&= [f_{MN}, f_{IJ}]_{KL}.
\end{aligned} \tag{4.21}$$

By the third equation of (4.19), the structure constants $\tilde{C}_{IJ,KL,MN}$ are indeed totally antisymmetric in the 3 pairs of indices. Therefore Eq. (4.18) now takes the following form

$$[f, C_{IJ}] = 0 \quad \text{or} \quad [f_{MN}, f_{IJ}]_{KL} + [f_{KL}, f_{IJ}]_{MN} = 0, \tag{4.22}$$

[7] This is a comment by E. Witten, quoted in the second paper of Ref. [15].

which is nothing but the third equation of Eq. (4.19). Namely both Eq. (4.15) and Eq. (4.18) can be written as the third equation of Eq. (4.19), if we express everything in terms of the 3-algebra structure constants f_{IJKL}.

Note that we use k_{mn} to lower an adjoint index, while use ω_{IJ} to lower a fundamental index. If Eq. (4.1) holds, then Eq. (2.7) implies a compatible condition between k_{mn} and ω_{IJ}. Eq. (2.7) is equivalent to $k_{nm}\tau_I^{mK}\omega_{KJ} + k_{nm}\tau_J^{mK}\omega_{IK} = 0$, i.e.

$$\tilde{\tau}_{nIJ} - k_{nm}\omega_{IK}\tau_J^{mK} = 0, \tag{4.23}$$

where $\tilde{\tau}_{nIJ} \equiv k_{nm}\tau_{IJ}^m$.

4.3 Three-Algebras and Embedding Tensors

In Ref. [29, 30], the authors derive some extended superconformal gauge theories by taking a conformal limit of $D = 3$ gauged supergravity theories. In their approach, the embedding tensor plays a crucial role. By definition, the embedding tensor $\theta_{mn} = \theta_{nm}$ acts as a projector [30]:

$$D_\mu = \partial_\mu - A_\mu^m \theta_{mn} t^n, \tag{4.24}$$

where t^n is a set of independent generators. The above equation says that θ_{mn} projects t^n onto another set of generators $\tilde{t}_m = \theta_{mn}t^n$, whose symmetries are gauged. Let us now consider the commutator

$$[\tilde{t}_m, \tilde{t}_n] = \theta_{mp}\theta_{ns}C^{ps}{}_q t^q. \tag{4.25}$$

Since we expect that $[\tilde{t}_m, \tilde{t}_n] = \tilde{C}_{mn}{}^r\tilde{t}_r$, we must set

$$\theta_{mp}\theta_{ns}C^{ps}{}_q = \tilde{C}_{mn}{}^r\theta_{rq}. \tag{4.26}$$

It is necessary to examine the Jacobi identity

$$\begin{aligned}
&[[\tilde{t}_m, \tilde{t}_n], \tilde{t}_p] + [[\tilde{t}_n, \tilde{t}_p], \tilde{t}_m] + [[\tilde{t}_p, \tilde{t}_m], \tilde{t}_n] \\
&= (\tilde{C}_{mn}{}^s\tilde{C}_{sp}{}^r + \tilde{C}_{np}{}^s\tilde{C}_{sm}{}^r + \tilde{C}_{pm}{}^s\tilde{C}_{sn}{}^r)\theta_{rq}t^q \\
&= (C^{lq}{}_r C^{rs}{}_t + C^{qs}{}_r C^{rl}{}_t + C^{sl}{}_r C^{rq}{}_t)\theta_{ml}\theta_{nq}\theta_{ps}t^t = 0.
\end{aligned} \tag{4.27}$$

In the last line we have used (4.26). The last line is nothing but the Jacobi identity satisfied by $C^{mn}{}_p$. So Eq. (4.27) is indeed the desired result. To construct a physical theory, the embedding tensor is required to be invariant under the transformations which are gauged. Since the embedding tensor θ_{mn} carries two adjoint indices, we have to set

$$\tilde{C}_{nq}{}^r\theta_{rs} + \tilde{C}_{ns}{}^r\theta_{qr} = 0. \tag{4.28}$$

Taking account of (4.26), the above equation is equivalent to

$$\theta_{np}\theta_{qm}C^{pm}{}_s + \theta_{np}\theta_{sm}C^{pm}{}_q = 0. \tag{4.29}$$

This quadratic constraint takes the same form for all extended supergravity theories. We will focus on the $\mathcal{N} = 5$ case. If we represent the adjoint index m as a pair of fundamental

indices IJ, the embedding tensor becomes $\theta_{IJ,KL}$, satisfying the same symmetry properties as f_{IJKL} do (see (2.16)) [29]. To construct $\mathcal{N}=5$ supergravity theories, the embedding tensor is required to satisfy the linear constraint:

$$\theta_{(IJ,K)L}=0, \tag{4.30}$$

and the structure constants in (4.29) are required to be those of $Sp(2L)$ [29]. We observe that if one identifies the embedding tensor θ_{mn} with the Killing-Cartan metric k_{mn}, Eq. (4.29) is precisely the same as Eq. (4.16), which is the FI satisfied by the 3-algebra structure constants $f_{IJKL}=k_{mn}\tau^m_{IJ}\tau^n_{KL}$. Recall that f_{IJKL} also play the role of the Killing-Cartan metric (see Sec. 4.2). So identifying the embedding tensor with the Killing-Cartan metric is equivalent to identifying the embedding tensor with the 3-algebra structure constants. With this identification, Eq. (4.30) is also solved since it is nothing but $f_{(IJK)L}=0$. We are therefore led to the conclusion that f_{IJKL} also play the role of the embedding tensor. It is straightforward to generalize the discussion of this section to the cases with other values of $\mathcal{N}$.

In summary, if we realize the symplectic 3-algebra in terms of the superalgebra (4.2), we find that $f_{IJKL}=k_{mn}\tau^m_{IJ}\tau^n_{KL}$ play four roles simultaneously:

- f_{IJKL} are the structure constants of the symplectic 3-algebra or the double graded commutator (4.4);
- f_{IJKL} furnish a quaternion representation of the bosonic part of the superalgebra;
- f_{IJKL} play the role of the Killing-Cartan metric;
- f_{IJKL} are the components of the embedding tensor used to construct the $D=3$ extended supergravity theories.

5. N=4, 5 Theories in Terms of Lie Algebras

The $\mathcal{N}=4,5$ theories in Sec. 2 and 3 are constructed in terms of 3-algebras. After the discussions of the last section, we are ready to derive their ordinary Lie Algebra constructions by the solution (4.1).

5.1 $\mathcal{N}=5$ Theories in Terms of Lie Algebras

With the solution

$$f_{IJKL}=k_{mn}\tau^m_{IJ}\tau^n_{KL},\quad [\tau^m,\tau^n]_{IJ}=C^{mn}{}_p\tau^p_{IJ}, \tag{5.1}$$

the gauge field becomes

$$\tilde{A}_\mu{}^K{}_L=A^{IJ}_\mu f_{IJ}{}^K{}_L=A^{IJ}_\mu\tau^m_{IJ}k_{mn}\tau^{nK}{}_L\equiv A^m_\mu k_{mn}\tau^{nK}{}_L. \tag{5.2}$$

Following Ref. [31], we define the ‘momentum map’ and ‘current ’ operator as follows

$$\mu^m_{AB}\equiv\tau^m_{IJ}Z^I_AZ^J_B,\quad j^m_{AB}\equiv\tau^m_{IJ}Z^I_A\psi^J_B. \tag{5.3}$$

Here $A = 1, \ldots, 4$ is the fundamental index of the R-symmetry group $Sp(4)$. Substituting the (5.1) and (5.2) into the Lagrangian (2.54) gives

$$\begin{aligned}\mathcal{L} = & \frac{1}{2}(-D_\mu \bar{Z}_I^A D^\mu Z_A^I + i\bar{\psi}_I^A \gamma_\mu D^\mu \psi_A^I) - \frac{i}{2}\omega^{AB}\omega^{CD}k_{mn}(j_{AC}^m j_{BD}^n - 2j_{AC}^m j_{DB}^n) \\ & + \frac{1}{2}\epsilon^{\mu\nu\lambda}(k_{mn}A_\mu^m \partial_\nu A_\lambda^n + \frac{1}{3}\tilde{C}_{mnp}A_\mu^m A_\nu^n A_\lambda^p) \\ & + \frac{1}{30}\tilde{C}_{mnp}\mu^{mA}{}_B \mu^{nB}{}_C \mu^{pC}{}_A + \frac{3}{20}k_{mp}k_{ns}(\tau^m\tau^n)_{IJ}Z^{AI}Z_A^J \mu^{pB}{}_C \mu^{sC}{}_B.\end{aligned} \tag{5.4}$$

Similarly, with the solution (5.1), the supersymmetry transformation law becomes

$$\begin{aligned}\delta Z_A^I &= i\epsilon_A{}^B \psi_B^I, \\ \delta\psi_A^I &= \gamma^\mu D_\mu Z_B^I \epsilon^B{}_A + \frac{1}{3}k_{mn}\tau^{mI}{}_J\omega^{BC}Z_B^J\mu_{CD}^n\epsilon^D{}_A - \frac{2}{3}k_{mn}\tau^{mI}{}_J\omega^{BD}Z_C^J\mu_{DA}^n\epsilon^C{}_B, \\ \delta A_\mu^m &= i\epsilon^{AB}\gamma_\mu j_{AB}^m.\end{aligned} \tag{5.5}$$

Here the parameter $\epsilon_A{}^B$ obeys the traceless condition and the reality condition (2.62). The $\mathcal{N} = 5$ Lagrangian (5.4) and supersymmetry transformation law (5.5) are in agreement with those given in Ref. [33], which were derived directly in terms of ordinary Lie algebra.

In section (4.1), we demonstrate that if the structure constants of the 3-algebra are specified as (5.1), then the Lie algebra of the gauge group generated by the 3-algebra is just the bosonic part of the superalgebra (4.2). The following classical super-Liealgebras:

$$U(M|N), \quad OSp(M|2N), \quad OSp(2|2N), \quad F(4), \quad G(3), \quad D(2|1;\alpha), \tag{5.6}$$

(with α a continuous parameter) are of the same form as that of the superalgebra (4.2). Therefore their bosonic parts can be selected to be the Lie algebras of the gauge groups of the $\mathcal{N} = 5$ theories. Especially, if we choose the $U(M|N)$ or $OSp(2|2N)$, whose bosonic part is in the two conjugate representations $(R \oplus \bar{R})$, then the supersymmetry will get enhanced to $\mathcal{N} = 6$ [33]. In the case of $OSp(M|2N)$, the theory has been conjectured to be the dual gauge theory of M2-branes in orbifold $\mathbf{C}^4/\hat{\mathbf{D}}_k$, with $\hat{\mathbf{D}}_k$ the binary dihedral group [33, 34]. The gravity dual of this theory has been investigated in Ref. [34].

5.2 $\mathcal{N} = 4$ GW Theory in Terms of Lie Algebras

Here we consider only the $\mathcal{N} = 4$ GW theory without the 'twisted' hyper multiplets, i.e., setting $\Phi_{\dot{A}}^{a'} = 0$. Then with the solution for structure constants of the 3-algebra given by

$$f_{abcd} = k_{mn}\tau_{ab}^m\tau_{cd}^n, \quad [\tau^m, \tau^n]_{ab} = C^{mn}{}_p\tau_{ab}^p, \tag{5.7}$$

which satisfy the FI's as well as appropriate constraints and symmetry conditions, the gauge fields of the GW theory become

$$\tilde{A}_\mu{}^c{}_d = A_\mu^{ab}f_{ab}{}^c{}_d = A_\mu^{ab}\tau_{ab}^m k_{mn}\tau^{nc}{}_d \equiv A_\mu^m k_{mn}\tau^{nc}{}_d. \tag{5.8}$$

Following Ref. [31], we define the 'momentum map' and 'current ' operators as follows

$$\mu_{AB}^m \equiv \tau_{ab}^m Z_A^a Z_B^b, \quad j_{A\dot{B}}^m \equiv \tau_{ab}^m Z_A^a \psi_{\dot{B}}^b. \tag{5.9}$$

With Eqs (5.7) $\sim$ (5.9), Eqs. (3.41) and (3.42) become the Lagrangian and the supersymmetry law of the GW theory in Ref. [31], respectively:

$$\mathcal{L} = \frac{1}{2}\epsilon^{\mu\nu\lambda}(k_{mn}A^m_\mu\partial_\nu A^n_\lambda + \frac{1}{3}\tilde{C}_{mnp}A^m_\mu A^n_\nu A^p_\lambda) + \frac{1}{2}(-D_\mu\bar{Z}^A_a D^\mu Z^a_A + i\bar{\psi}^{\dot{A}}_a\gamma^\mu D_\mu\psi^a_{\dot{A}})$$
$$-\frac{i}{2}k_{mn}j^m_{A\dot{B}}j^{nA\dot{B}} - \frac{1}{24}\tilde{C}_{mnp}\mu^{mA}{}_B\mu^{nB}{}_C\mu^{pC}{}_A, \tag{5.10}$$

with $\tilde{C}_{mnp} = k_{mr}k_{ns}C^{rs}{}_p$ and

$$\delta Z^a_A = i\epsilon_A{}^{\dot{A}}\psi^a_{\dot{A}},$$
$$\delta\psi^a_{\dot{A}} = -\gamma^\mu D_\mu Z^a_B\epsilon^{\dagger}_{\dot{A}}{}^B - \frac{1}{3}k_{mn}\tau^{ma}{}_b Z^b_B\mu^{nB}{}_C\epsilon^{\dagger}_{\dot{A}}{}^C,$$
$$\delta A^m_\mu = i\epsilon^{A\dot{B}}\gamma_\mu j^m_{A\dot{B}}. \tag{5.11}$$

Since we derived the GW theory by decomposing the $\mathcal{N}=5$ theory and setting the twisted mulitplets to zero, the classical superalgebras, that are used to realize the 3-algebra, must be the same as those used in the $\mathcal{N}=5$ case, i.e.

$$U(M|N),\quad OSp(M|2N),\quad OSp(2|2N),\quad F(4),\quad G(3),\quad D(2|1;\alpha). \tag{5.12}$$

Indeed, they are of the same form as that of the superalgebra (4.2). Therefore their bosonic parts can be selected to be the Lie algebras of the gauge groups of the GW theory; and the corresponding representations are determined by the fermionic generators.

For the cases [32, 33] with both un-twisted and twisted hyper-multiplets, a pair of super Lie algebras are needed, which were discussed in a representation theory approach in Ref. [35]. Since the situation is much more complicated, we leave the presentation of these cases in terms of ordinary Lie algebras, as well as their generalizations, within our superspace and super Lie algebra approach to a subsequent paper [44].

6. Conclusions

In this paper, we have combined the symplectic 3-algebra with the superspace formalism by letting the matter superfields take values in the symplectic 3-algebra. Based on the 3-algebra, we then have constructed the general $\mathcal{N}=5$ CMS theory by enhancing the $\mathcal{N}=1$ supersymmetry to $\mathcal{N}=5$. The $\mathcal{N}=5$ Lagrangian is same as the one derived with an on-shell approach [38].

We have constructed the general $\mathcal{N}=4$ CSM theory by decomposing one $\mathcal{N}=5$ hypermultiplet into a $\mathcal{N}=4$ un-twisted hypermultiplet and a $\mathcal{N}=4$ twisted hypermultiplet, and then proposing a new superpotential. In deriving the general $\mathcal{N}=4$ CSM theory, we have also decomposed the set of 3-algebra generators into two sets of 3-algebra generators. As a result, both the FI's and 3-brackets are decomposed into 4 sets. The resulting general $\mathcal{N}=4$ CSM theory is a quiver gauge theory based on the 3-algebra. We have also examined the closure of the $\mathcal{N}=4$ algebra.

We then have realized the symplectic 3-algebra in terms of the super Lie algebra (4.2). The 3-bracket is realized in terms of a double graded bracket: $[T_I, T_J; T_K] \doteq$

$[\{Q_I, Q_J\}, Q_K]$, where Q_I are the fermionic generators; the structure constants of the 3-algebra are just the structure constants of the double graded bracket, i.e., $f_{IJKL} = k_{mn}\tau^m_{IJ}\tau^n_{KL}$. The fundamental identity of the 3-algebra is equivalent to the MMQ Jacobi identity of the super Lie algebra, where M's are the bosonic generators in the super Lie algebra. The linear constraint equation $f_{(IJK)L} = 0$, required by the enhancement of the supersymmetry, is equivalent to the QQQ Jacobi identity.

We have also analyzed the relations between the symplectic 3-algebra and the ordinary Lie algebra. The fundamental identity of 3-algebra can be solved in terms of a tensor product: $f_{IJKL} = k_{mn}\tau^m_{IJ}\tau^n_{KL}$. We have proved that the structure constants f_{IJKL} furnish a quaternion representation of the bosonic part of the super Lie algebra (4.2), and f_{IJKL} also play a role of Killing-Cartan metric. We found that the FI of the 3-algebra can be converted into an ordinary commutator (4.19); the structure constants of the commutator are (4.21). The FI of the 3-algebra can be understood as the statement that the structure constants of the commutator (4.21) are total antisymmetric (see Eqs. (4.22)).

We have proved that the components of an embedding tensor [29, 30], used to construct the $D = 3$ extended supergravity theories, are just the structure constants of the 3-algebra. Hence the concepts and techniques of the 3-algebra may be used to construct new $D = 3$ extended supergravity theories.

The general $\mathcal{N} = 5$ CSM theories and the $\mathcal{N} = 4$ GW CSM theories in terms of ordinary Lie algebras are rederived, respectively, in our superspace approach. The presentation of general $\mathcal{N} = 4$ CSM theories is left for a subsequent paper [44]. In this way, we have been able to derive all known $\mathcal{N} = 4, 5$ superconformal Chern-Simons matter theories, as well as some new $\mathcal{N} = 4$ quiver gauge theories (to be presented in [44]). Thus our superspace formulation for the super-Lie-algebra realization of symplectic 3-algebras provides a unified treatment of all known $\mathcal{N} = 4, 5, 6, 8$ CSM theories, including new examples of $\mathcal{N} = 4$ quiver gauge theories as well.

The extended ($\mathcal{N} \geq 4$) CSM theories can be also constructed by using $\mathcal{N} = 2, 3$ superspace formulations in an ordinary *Lie (2-)algebra* approach [27, 46, 47]. The $\mathcal{N} = 2, 3$ superspace formulations are more restrictive than the $\mathcal{N} = 1$ formulation; hence the calculations may be simplified, and it may be easier to enhance the supersymmetry from $\mathcal{N} = 2$ or $\mathcal{N} = 3$ to $\mathcal{N} = 6, 8$. It would be nice to construct the extended CSM theories by using the $\mathcal{N} = 2, 3$ superspace formulations in a *3-algebra* approach.

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A. Conventions and Useful Identities

A.1 Spinor Algebra

In $1+2$ dimensions, the gamma matrices are defined as

$$(\gamma_\mu)_\alpha{}^\gamma(\gamma_\nu)_\gamma{}^\beta + (\gamma_\nu)_\alpha{}^\gamma(\gamma_\mu)_\gamma{}^\beta = 2\eta_{\mu\nu}\delta_\alpha{}^\beta. \tag{A.1}$$

For the metric we use the $(-,+,+)$ convention. The gamma matrices in the Majorana representation can be defined in terms of Pauli matrices: $(\gamma_\mu)_\alpha{}^\beta = (i\sigma_2, \sigma_1, \sigma_3)$, satisfying the important identity

$$(\gamma_\mu)_\alpha{}^\gamma(\gamma_\nu)_\gamma{}^\beta = \eta_{\mu\nu}\delta_\alpha{}^\beta + \varepsilon_{\mu\nu\lambda}(\gamma^\lambda)_\alpha{}^\beta. \tag{A.2}$$

We also define $\varepsilon^{\mu\nu\lambda} = -\varepsilon_{\mu\nu\lambda}$. So $\varepsilon_{\mu\nu\lambda}\varepsilon^{\rho\nu\lambda} = -2\delta_\mu{}^\rho$. We raise and lower spinor indices with an antisymmetric matrix $\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta}$, with $\epsilon_{12} = -1$. For example, $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$ and $\gamma^\mu_{\alpha\beta} = \epsilon_{\beta\gamma}(\gamma^\mu)_\alpha{}^\gamma$, where ψ_β is a Majorana spinor. Notice that $\gamma^\mu_{\alpha\beta} = (\mathbb{1}, -\sigma^3, \sigma^1)$ are symmetric in $\alpha\beta$. A vector can be represented by a symmetric bispinor and vice versa:

$$A_{\alpha\beta} = A_\mu\gamma^\mu_{\alpha\beta}, \qquad A_\mu = -\frac{1}{2}\gamma_\mu^{\alpha\beta}A_{\alpha\beta}. \tag{A.3}$$

We use the following spinor summation convention:

$$\psi\chi = \psi^\alpha\chi_\alpha, \qquad \psi\gamma_\mu\chi = \psi^\alpha(\gamma_\mu)_\alpha{}^\beta\chi_\beta, \tag{A.4}$$

where ψ and χ are anti-commuting Majorana spinors. In $1+2$ dimensions the Fierz transformations are

$$\begin{aligned}
(\lambda\chi)\psi &= -\frac{1}{2}(\lambda\psi)\chi - \frac{1}{2}(\lambda\gamma_\nu\psi)\gamma^\nu\chi, \qquad (A.5)\\
(\psi_1\psi_2)(\psi_3\psi_4) &= (\psi_1\psi_2)(\psi_4\psi_3) = -\frac{1}{2}(\psi_1\psi_3)(\psi_4\psi_2) - \frac{1}{2}(\psi_1\gamma_\nu\psi_3)(\psi_4\gamma^\nu\psi_2),\\
(\psi_1\gamma_\mu\psi_2)(\psi_3\psi_4) &= -\frac{1}{2}(\psi_1\gamma_\mu\psi_3)(\psi_4\psi_2) - \frac{1}{2}(\psi_1\psi_3)(\psi_4\gamma_\mu\psi_2) + \frac{1}{2}\varepsilon_{\mu\nu\lambda}(\psi_1\gamma^\nu\psi_3)(\psi_4\gamma^\lambda\psi_2).
\end{aligned}$$

A.2 The $\mathcal{N}=1$ Superspace

In this subsection, we mainly follow the conventions of Ref. [33]. We denote the superspace coordinates as θ^α. A real scalar superfield Φ can be expanded as

$$\Phi = \phi + i\theta\psi - \frac{i}{2}\theta^2F, \tag{A.6}$$

where θ and ψ are Majorana spinors. The superalgebra

$$\{Q_\alpha, Q_\beta\} = -2\gamma^\mu_{\alpha\beta}P_\mu \tag{A.7}$$

can be realized in terms of superspace derivatives:

$$Q_\alpha = i\partial_\alpha + \theta^\beta\partial_{\beta\alpha}. \tag{A.8}$$

The super-covariant derivative must anti-commute with Q_α; it takes the following form:

$$\mathscr{D}_\alpha = \partial_\alpha + i\theta^\beta \partial_{\beta\alpha}. \tag{A.9}$$

The supersymmetry transformation of Φ is defined as

$$\delta\Phi = -i\epsilon^\alpha Q_\alpha \Phi \equiv \delta\phi + i\theta\delta\psi - \frac{i}{2}\theta^2\delta F. \tag{A.10}$$

Equating powers of θ^α gives the supersymmetry transformations of the component fields:

$$\begin{aligned} \delta\phi &= i\epsilon^\alpha \psi_\alpha, & \text{(A.11)} \\ \delta\psi_\alpha &= -\partial_\alpha{}^\beta \phi \epsilon_\beta - F\epsilon_\alpha, & \text{(A.12)} \\ \delta F &= i\epsilon^\alpha \partial_\alpha{}^\beta \psi_\beta. & \text{(A.13)} \end{aligned}$$

In the Wess-Zumino gauge, the superconnection becomes

$$\Gamma_\alpha = i\theta^\beta A_{\alpha\beta} + \theta^2 \chi_\alpha, \tag{A.14}$$

and the supersymmetry transformations for the component fields are

$$\begin{aligned} \delta A_\mu &= -i\epsilon^\alpha (\gamma_\mu)_\alpha{}^\beta \chi_\beta, & \text{(A.15)} \\ \delta\chi_\alpha &= -\frac{1}{2} F_{\mu\nu} (\gamma^{\mu\nu})_\alpha{}^\beta \epsilon_\beta. & \text{(A.16)} \end{aligned}$$

The Berezin integral is defined as

$$\int d^2\theta \theta^2 = -4. \tag{A.17}$$

The superpotential is given by

$$\mathcal{L}_W = \frac{i}{2} \int d^2\theta W(\Phi) = -\frac{i}{2} W''(\phi)\psi^2 - W'(\phi)F. \tag{A.18}$$

A.3 $SU(2) \times SU(2)$ Identities

We define the 4 sigma matrices as

$$\sigma^a{}_A{}^{\dot{B}} = (\sigma^1, \sigma^2, \sigma^3, i\mathbb{1}), \tag{A.19}$$

by which one can establish a connection between the $SU(2) \times SU(2)$ and $SO(4)$ group. These sigma matrices satisfy the following Clifford algebra:

$$\begin{aligned} \sigma^a{}_A{}^{\dot{C}} \sigma^{b\dagger}{}_{\dot{C}}{}^B + \sigma^b{}_A{}^{\dot{C}} \sigma^{a\dagger}{}_{\dot{C}}{}^B &= 2\delta^{ab} \delta_A{}^B, & \text{(A.20)} \\ \sigma^{a\dagger}{}_{\dot{A}}{}^C \sigma^b{}_C{}^{\dot{B}} + \sigma^{b\dagger}{}_{\dot{A}}{}^C \sigma^a{}_C{}^{\dot{B}} &= 2\delta^{ab} \delta_{\dot{A}}{}^{\dot{B}}. & \text{(A.21)} \end{aligned}$$

We use antisymmetric matrices

$$\epsilon_{AB} = -\epsilon^{AB} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon_{\dot{A}\dot{B}} = -\epsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.22}$$

to raise or lower un-dotted and dotted indices, respectively. For example, $\sigma^{a\dagger\dot{A}B} = \epsilon^{\dot{A}\dot{B}}\sigma^{a\dagger}{}_{\dot{B}}{}^{B}$ and $\sigma^{aB\dot{A}} = \epsilon^{BC}\sigma^{a}{}_{C}{}^{\dot{A}}$. The sigma matrix σ^a satisfies a reality condition

$$\sigma^{a\dagger}{}_{\dot{A}}{}^{B} = -\epsilon^{BC}\epsilon_{\dot{A}\dot{B}}\sigma^{a}{}_{C}{}^{\dot{B}}, \quad \text{or} \quad \sigma^{a\dagger\dot{A}B} = -\sigma^{aB\dot{A}}. \tag{A.23}$$

The antisymmetric matrix ϵ_{AB} satisfies an important identity

$$\epsilon_{AB}\epsilon^{CD} = -(\delta_{A}{}^{C}\delta_{B}{}^{D} - \delta_{A}{}^{D}\delta_{B}{}^{C}), \tag{A.24}$$

and $\epsilon_{\dot{A}\dot{B}}$ satisfies a similar identity.

Define

$$\sigma^{A\dot{B}} \equiv c_a\sigma^{aA\dot{B}} \quad \text{and} \quad c_a c^a = 1, \tag{A.25}$$

where c_a are real coefficients, then the following identity holds

$$\sigma^{A\dot{C}}\sigma^{B\dot{D}} - \sigma^{A\dot{D}}\sigma^{B\dot{C}} = \epsilon^{AB}\epsilon^{\dot{C}\dot{D}}. \tag{A.26}$$

This identity is useful when we construct the $\mathcal{N} = 4$ theory.

Define the parameter for the $\mathcal{N} = 4$ supersymmetry transformations as $\epsilon^{A\dot{B}} = \epsilon_a\sigma^{aA\dot{B}}$. The following identities are useful in checking the closure of the $\mathcal{N} = 4$ superalgebra:

$$\begin{aligned} i(\epsilon_1^{A\dot{C}}\epsilon^{\dagger}_{2\dot{C}}{}^{B} - \epsilon_2^{A\dot{C}}\epsilon^{\dagger}_{1\dot{C}}{}^{B}) &\equiv u^{AB} = u^{BA}, \\ i(\epsilon_1^{\dagger\dot{A}C}\epsilon_{2C}{}^{\dot{B}} - \epsilon_2^{\dagger\dot{A}C}\epsilon_{1C}{}^{\dot{B}}) &\equiv u^{\dot{A}\dot{B}} = u^{\dot{B}\dot{A}}, \\ i(\epsilon_{1A}{}^{\dot{A}}\gamma^{\mu}\epsilon^{\dagger}_{2\dot{A}}{}^{B} - \epsilon_{2A}{}^{\dot{A}}\gamma^{\mu}\epsilon^{\dagger}_{1\dot{A}}{}^{B}) &= i\epsilon_1^{C\dot{C}}\gamma^{\mu}\epsilon_{2C\dot{C}}\delta_{A}{}^{B} \equiv v^{\mu}\delta_{A}{}^{B}, \\ 2(\epsilon_{1A\dot{A}}\epsilon^{\dagger}_{2\dot{B}B} - \epsilon_{2A\dot{A}}\epsilon^{\dagger}_{1\dot{B}B}) &= (\epsilon_{1A}{}^{\dot{C}}\epsilon^{\dagger}_{2\dot{C}B} - \epsilon_{2A}{}^{\dot{C}}\epsilon^{\dagger}_{1\dot{C}B})\epsilon_{\dot{A}\dot{B}} + (\epsilon^{\dagger}_{1\dot{B}}{}^{C}\epsilon_{2C\dot{A}} - \epsilon^{\dagger}_{2\dot{B}}{}^{C}\epsilon_{1C\dot{A}})\epsilon_{AB}, \\ i\epsilon_{AB}\epsilon_{\dot{C}\dot{D}}\epsilon_1^{E\dot{E}}\gamma^{\mu}\epsilon_{2E\dot{E}} &= i(\epsilon_{1B\dot{C}}\gamma^{\mu}\epsilon^{\dagger}_{2\dot{D}A} - \epsilon_{2B\dot{C}}\gamma^{\mu}\epsilon^{\dagger}_{1\dot{D}A}) - i(\epsilon_{1A\dot{C}}\gamma^{\mu}\epsilon^{\dagger}_{2\dot{D}B} - \epsilon_{2A\dot{C}}\gamma^{\mu}\epsilon^{\dagger}_{1\dot{D}B}). \end{aligned} \tag{A.27}$$

A.4 $SO(5)$ Gamma Matrices

In this subsection, in order to avoid introducing too many indices into the theory, we still use the capital letters $A, B, \ldots$ to label the $Sp(4)$ indices. However, now the index A runs from 1 to 4. (In Sec. A.3, the indices A and $\dot{B}$ run from 1 to 2.) We hope this does not cause any confusion.

Since $Sp(4) \cong SO(5)$, it is useful to introduce the $SO(5)$ gamma matrices. We define the $SO(5)$ gamma matrices as

$$\gamma^{a}_{A}{}^{B} = \begin{pmatrix} 0 & \sigma^a \\ \sigma^{a\dagger} & 0 \end{pmatrix}, \qquad \gamma^{5}_{A}{}^{B} = (\gamma^1\gamma^2\gamma^3\gamma^4)_{A}{}^{B}, \tag{A.28}$$

where σ^a are defined by (A.19). Notice that γ^{mB}_{A} $(m = 1, \ldots, 5)$ are Hermitian, satisfying the Clifford algebra

$$\gamma^{mC}_{A}\gamma^{n}_{C}{}^{B} + \gamma^{nC}_{A}\gamma^{mB}_{C} = 2\delta^{mn}\delta_{A}{}^{B}. \tag{A.29}$$

We use an antisymmetric matrix $\omega_{AB} = -\omega^{AB}$ to lower and raise indices; for instance

$$\gamma^{mAB} = \omega^{AC}\gamma^{mB}_{C}. \tag{A.30}$$

It can be chosen as the charge conjugate matrix:

$$\omega^{AB} = \begin{pmatrix} \epsilon^{AB} & 0 \\ 0 & \epsilon^{\dot{A}\dot{B}} \end{pmatrix}. \tag{A.31}$$

(Recall that A and $\dot{B}$ of the RHS run from 1 to 2.)

By the definition (A.28) and the convention (A.30), the gamma matrix γ^m is antisymmetric and traceless, and satisfies a reality condition

$$\gamma^{mAB} = -\gamma^{mBA} \quad , \quad \gamma^{mA}_{A} = 0 \quad \text{and} \quad \gamma^{m*}_{AB} = \gamma^{mAB} = \omega^{AC}\omega^{BD}\gamma^{m}_{CD}. \tag{A.32}$$

The $Sp(4)$ generators are defined as

$$\Sigma^{mnB}_{A} = \frac{1}{4}[\gamma^m, \gamma^n]_A{}^B. \tag{A.33}$$

There is a useful $Sp(4)$ identity

$$\varepsilon^{ABCD} = -\omega^{AB}\omega^{CD} + \omega^{AC}\omega^{BD} - \omega^{AD}\omega^{BC}. \tag{A.34}$$

B. Verification of $Sp(4)$ Global symmetry of the $\mathcal{N} = 5$ Bosonic Potential

In this section we will prove that the bosonic potential (2.52) has an $Sp(4)$ global symmetry. For convenience, we cite it here:

$$\begin{aligned} -V = &\frac{1}{18} f_{IJKO} f^{O}{}_{LMN}(-\omega^{AC}\omega^{BE}\omega^{DF} + 2\omega^{AC}\gamma^{BE}\gamma^{DF} \\ &+2\omega^{DF}\gamma^{AC}\gamma^{BE} - 4\omega^{BE}\gamma^{AC}\gamma^{DF}) Z^I_A Z^J_B Z^K_C Z^L_D Z^M_E Z^N_F. \end{aligned} \tag{B.1}$$

It can be seen that the first term is manifestly $Sp(4)$ invariant. So we need only to consider the last three terms. Denote them as $-V'$. For $-V'$, the part proportional to $Z^{(K}_{(C} Z^{L)}_{D)}$ vanishes by the FI (2.5), so the remaining part of $-V'$ is

$$\begin{aligned} -V'_A &= \frac{2}{9}(\omega^{AC}\gamma^{BE}\gamma^{DF} - \omega^{BE}\gamma^{AC}\gamma^{DF}) f_{IJKO} f^{O}{}_{LMN} Z^I_A Z^J_B Z^{[K}_{[C} Z^{L]}_{D]} Z^M_E Z^N_F \\ &\equiv \frac{2}{9}(P_1 - P_2). \end{aligned} \tag{B.2}$$

On the other hand, by using the constraint condition $f_{(IJK)O} = 0$ (see (2.47)) and the FI (2.5), one can rewrite (B.2) as

$$\begin{aligned} -V'_A &= \frac{1}{9}(\omega^{AC}\gamma^{BE}\gamma^{DF} - \omega^{BE}\gamma^{AC}\gamma^{DF} + \omega^{CD}\gamma^{AE}\gamma^{BF} - \omega^{BE}\gamma^{CD}\gamma^{AF}) \\ &\quad \times f_{IJKO} f^{O}{}_{LMN} Z^I_A Z^J_B Z^{[K}_{[C} Z^{L]}_{D]} Z^M_E Z^N_F \\ &\equiv \frac{1}{9}(P_1 - P_2 + P_3 - P_4). \end{aligned} \tag{B.3}$$

Comparing (B.2) with (B.3) gives

$$P_1 - P_2 = P_3 - P_4. \tag{B.4}$$

We observe that $2P_2 + P_4$ is an $Sp(4)$ invariant quantity:

$$
\begin{aligned}
2P_2 + P_4 &= (2\omega^{BE}\gamma^{A[C}\gamma^{D]F} + \omega^{BE}\gamma^{CD}\gamma^{AF}) f_{IJKO} f^{O}{}_{LMN} Z^I_A Z^J_B Z^{[K}_{[C} Z^{L]}_{D]} Z^M_E Z^N_F \\
&= \omega^{BE}\varepsilon^{ACDF} f_{IJKO} f^{O}{}_{LMN} Z^I_A Z^J_B Z^{[K}_{[C} Z^{L]}_{D]} Z^M_E Z^N_F \\
&\equiv I.
\end{aligned} \tag{B.5}
$$

In the second line we have used the key identity (2.38). By using the second line of (2.38), i.e. $\varepsilon^{ABCD} = -\omega^{AB}\omega^{CD} + \omega^{AC}\omega^{BD} - \omega^{AD}\omega^{BC}$, we find that I can be written as

$$
-I = \varepsilon_G{}^{ACE}\varepsilon^{GBDF} f_{IJKO} f^{O}{}_{LMN} Z^I_A Z^J_B Z^{[K}_{[C} Z^{L]}_{D]} Z^M_E Z^N_F. \tag{B.6}
$$

On the other hand, substituting the first line of (2.38) ($-\varepsilon^{ABCD} = \gamma^{AC}\gamma^{BD} - \gamma^{BC}\gamma^{AD} + \gamma^{BA}\gamma^{CD}$) into the RHS of (B.6), we obtain

$$
-I = 4P_1 - 2P_2 + P_3. \tag{B.7}
$$

Combining (B.4), (B.5) and (B.7), we find that

$$
P_1 - P_2 = -\frac{2}{5} I. \tag{B.8}
$$

Substituting the above equation into Eq. (B.2), we reach the desired result:

$$
-V' = -\frac{4}{45} I. \tag{B.9}
$$

Recall that we denote the last three terms of (B.1) as $-V'$, so the bosonic potential (B.1) is indeed $Sp(4)$ invariant. After some work, we reach the final expression for the bosonic potential (B.1):

$$
-V = \frac{1}{60}(2f_{IJK}{}^{O} f_{OLMN} - 9f_{KLI}{}^{O} f_{ONMJ} + 2f_{IJL}{}^{O} f_{OKMN}) Z^N_A Z^{AI} Z^J_B Z^{BK} Z^L_C Z^{CM}. \tag{B.10}
$$

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